JOURNAL OF NUMBER THEORY **21**, 44–80 (1985)

Class Field Theory for Curves over Local Fields

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Communicated by Y. Ihara

Received June 10, 1983

The purpose of this paper is to generalize some results of Bloch [3] concerning class field theory for curves over local fields. Bloch developed his theory mainly under the assumption that curves have good reduction. By using a different method, we shall develope a theory including the bad reduction case.

INTRODUCTION

Let k be a complete discrete valuation field with finite residue field and X a projective smooth geometrically irreducible curve over k. In this paper, we shall study in detail the structure of the abelian fundamental group $\pi_1^{ab}(X)$ of X using the group $V(X)$ introduced by Bloch [3] by K-theoretic method. Its outline is as follows: First, there is an exact sequence (Chap. II (3.2));

$$0 \to (T)_G \to \pi_1^{ab}(X) \to \mathrm{Gal}(k^{ab}/k) \to 0,$$

where T is the Tate module of the Jacobian of X and $(T)_G$ is its coinvariant by $G = \mathrm{Gal}(k^{sep}/k)$. The group $\mathrm{Gal}(k^{ab}/k)$ corresponds to etale coverings of X coming from abelian extension of k, and its structure is already known by the local class field theory for k. Thus, the study of $\pi_1^{ab}(X)$ is mainly reduced to analyring the structure of $(T)_G$. In our study the most important role will be played by two "reciprocity maps"

$$\sigma\colon SK_1(X) \to \pi_1^{ab}(X),$$

$$\tau\colon V(X) \to (T)_G.$$

To define $SK_1(X)$ and $V(X)$, let K be the function field of X, P the set of all

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closed points of X, $\kappa(\mathfrak{p})$ ($\mathfrak{p} \in P$) the residue field of $\mathfrak{p}$ and $\mathrm{ord}_\mathfrak{p}$ the normalized additive valuation of K defined by $\mathfrak{p}$. Define the group

$$SK_1(X) = \mathrm{Coker}\left(\bigoplus_{\mathfrak{p} \in P} \partial_\mathfrak{p} \colon K_2(K) \to \bigoplus_{\mathfrak{p} \in P} \kappa(\mathfrak{p})^\times \right),$$

where $\partial_\mathfrak{p} \colon K_2(K) \to \kappa(\mathfrak{p})^\times$ is the boundary map in K-theory defined, for $\{f, g\} \in K_2(K)$, by

$$\partial_\mathfrak{p}\{f, g\} = (-1)^{\mathrm{ord}_\mathfrak{p}(f)\mathrm{ord}_\mathfrak{p}(g)} f^{\mathrm{ord}_\mathfrak{p}(g)} g^{-\mathrm{ord}_\mathfrak{p}(f)}|_\mathfrak{p}.$$

The pointwise norm maps $\kappa(\mathfrak{p})^\times \to k^\times$ for all $\mathfrak{p} \in P$ induce

$$\mathrm{Norm} \colon SK_1(X) \to k^\times,$$

and $V(X)$ is defined to be the kernel of this map Norm. Then we can construct two reciprocity maps σ and τ which fit into the commutative diagram

$$\begin{array}{ccccccc}
0 \to & V(X) & \to & SK_1(X) & \xrightarrow{\mathrm{Norm}} & k^\times & \\
& \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} & & \downarrow & \\
0 \to & (T)_G & \to & \pi_1^{\mathrm{ab}}(X)(X) & \to & \mathrm{Gal}(k^{\mathrm{ab}}/k) & \to 0,
\end{array}$$

where the right vertical arrow comes from the local class field theory for k. Now our main results are as follows. For simplicity of the statements, we suppose $\mathrm{ch}(k) = 0$ (when $\mathrm{ch}(k) = p \geqslant 0$, the restriction to "the prime-to-p part" is necessary in the following statements except (3)).

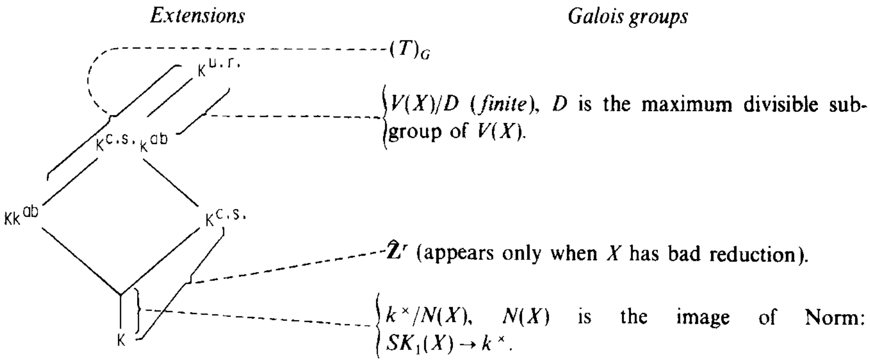
THEOREM. (1) (Chap. II (5.1)). *The kernel of σ (resp. τ) is the maximum divisible subgroup of $SK_1(X)$ (resp. $V(X)$).*

(2) (Chap. II (4.1)). *The image of τ is finite.*

(3) (Chap. II (2.6) and (3.5)). *The quotient of $\pi_1^{\mathrm{ab}}(X)$ by the closure of the image of σ and the cokernel of τ are both isomorphic to $\hat{\mathbf{Z}}^r$ with some common integer $r \geqslant 0$. ($\hat{\mathbf{Z}}$ denotes the completion of $\mathbf{Z}$ for the topology of subgroups of finite index.)*

Moreover, if we take an appropriate model $\mathscr{X}$ of X over $\mathrm{Spec}\, \mathcal{O}_k$, we can compute r explicitly from the "graph" of the special fiber of $\mathscr{X}$. This invariant attached to X will be called "the rank of X over k" (cf. Chap. II (2.5)).

These results give an explicit description of the structure of $(T)_G$ as an abstruct group. The Galois theoretic interpretation of our results is explained briefly by the following diagram, in which each extension is an Galois extention.



Here, $K^{\text{u.r.}}$ is the maximum abelian unramified extension of K (the function field of X) and $K^{\text{c.s.}}$ is the maximum subfield of $K^{\text{u.r.}}$ in which all $\mathfrak{p} \in P$ split completely. By definition, we have isomorphisms $\mathrm{Gal}(K^{\text{u.r.}}/K) \simeq \pi_1^{\text{ab}}(X)$ and $\mathrm{Gal}(K^{\text{u.r.}}/Kk^{\text{ab}}) \simeq (T)_G$. The extension $Kk^{\text{ab}} \cap K^{\text{c.s.}}$ over K is a finite (abelian) extension with the Galois group $k^\times/N(X)$ ($N(X)$ is the image of Norm: $SK_1(X) \to k^\times$), and it is trivial when X has a k-rational point.

Now we explain how we can view our theory as a special case in a program of generalized class field theory for varieties over higher local fields. Let k be any field and X a proper smooth variety over k. We define, for an integer $n \geqslant 0$,

$$SK_n(X) = \mathrm{Coker}\left(\partial\colon \bigoplus_{y \in X_1} K_{n+1}^M(\kappa(y)) \to \bigoplus_{x \in X_0} K_n^M(\kappa(x))\right),$$

where X_i ($i \geqslant 0$) denotes the set of all scheme-theoretic points $x \in X$ such that the closure $\overline{\{x\}}$ of $\{x\}$ is a scheme of dimension i, $\kappa(x)$ ($x \in X_i$) is the residue field of x, K_*^M are the Milnor K-groups and ∂ is induced by the boundary maps in Milnor K-theory. When k is an n-dimensional local field in the sense of Kato [13], the higher local field theory of Kato [13] allows us to construct a canonical homomorphism

$$\sigma\colon SK_n(X) \to \pi_1^{\text{ab}}(X),$$

which we call "the generalized reciprocity map for X." We expect that it will play an important role in the class field theory for varieties over n-dimensional local fields. In case $n = 0$ (i.e., k is finite) and $\dim X = 1$, this is nothing other than the reciprocity map in the classical class field theory for curves over finite fields (in fact, $SK_0(X)$ is equal to the group of divisor classes on X when $\dim X = 1$). The generalization of the classical theory (the case $n = 0$ and $\dim X = 1$) to the higher dimensional case (the case

$n=0$ and $\dim X$ is arbitrary) is established in [15], using the generalized reciprocity map for X. Our class field theory for curves over local fields corresponds to the case $n=1$ and $\dim X=1$, and one of the most notable differences of the theory for $n \geqslant 1$ from that for $n=0$ is the existence of nontrivial etale coverings of X in which all closed point split completely. Thus, in case $n=0$ the reciprocity map σ always has a dense image, but in case $n \geqslant 1$ this is not necessarily true (cf. Theorem (3)).

Finally, we explain the relation between our results and the results of Bloch [3]. Theorem (2) and Theorem (3) are generalizations of Bloch [3, (2.9) and (2.4)]; he proved the assertions when X has good reduction over k. In other words, under the assumption, he proved the surjectivity of the map $\tau\colon V(X) \to (T)_G$ and the finiteness of $(T)_G$. These generalizations enable us to complete *the unramified class field theory of 2-dimensional scheme $\mathscr{X}$ which is proper flat over the ring of integers of an algebraic number field* in [15]. This global result improves a previous work of Bloch [3], where he obtained a complete result under the assumption that $\mathscr{X}$ is proper *smooth* over the base scheme.

Our method is completely different from Bloch's. First, in the proof of Theorem 1, a result of Mercuriev and Suslin [18] plays an essential role. Secondly, the main tools used in the proof of Theorem 2 are a theorem of Miki [17, Chap. II (4.8)] and the higher local class field theory of Kato [1 and 2]. Finally, the key to the proof of Theorem 3 is the reciprocity law for 2-dimensional complete local rings, which is due to K. Kato and whose proof will be given in Chapter I.

0. Notations

For a commutative ring R, Ω^1_R denotes the absolute differential module of R over $\mathbf{Z}$ and Ω^q_R denotes the qth exterior power over R of Ω^1_R for each integer $q>0$.

If k is a field, k^{sep} (resp. $\bar{k}$) denotes the separable (resp. algebraic) closure of k and k^{ab} denotes the maximum abelian extension of k. If k is a discrete valuation field, ord_k denotes the normalized additive valuation of k, and $\mathcal{O}_k=\{x \in k \mid \mathrm{ord}_k(x) \geqslant 0\}$, $m_k=\{x \in k \mid \mathrm{ord}_k(x) \geqslant 1\}$, and $U^n_k=\{x \in k^\times \mid \mathrm{ord}_k(x-1) \geqslant n\}$ for each integer $n \geqslant 1$.

"A local field" means a complete discrete valuation field with *finite* residue field. "A two dimensional local field" means a complete discrete valuation field whose residue field is a local field.

For a scheme X and integer $m>0$, we identify the (etale) cohomology group $H^1(X, \mathbf{Z}/m\mathbf{Z})$ with the group of all continuous homomorphisms $\pi_1^{\mathrm{ab}}(X) \to \mathbf{Z}/m\mathbf{Z}$. When m is invertible in $\mathcal{O}_X$, μ_m denotes the sheaf of mth roots of unity on $X_{\acute{e}t}$ and $\mu_m^{\otimes r}$ denotes the rth exterior product of μ_m for each integer $r>0$.

For a scheme X and a prime number l, "a $\mathbf{Z}_l$-covering (resp. $\mathbf{Z}_l$-etale covering) of X" means a projective system $Y=\{Y_n\}_{n\in\mathbb{N}}$ of cyclic (resp. cyclic etale) coverings Y_n of degree l^n of X. We denote by $H^1(X, \mathbf{Z}_l)$ the group $\varprojlim_n H^1(X, \mathbf{Z}/l^n\mathbf{Z})$. By definition it is identified with the group of all continuous homomorphisms $\pi_1^{\mathrm{ab}}(X)\to\mathbf{Z}_l$.

For a profinite group G and a set S of prime numbers, we define "the S-part of G" (denoted by $G(S)$) to be $\varprojlim_N G/N$, where N runs over all open normal subgroups of G such that all prime divisors of $[G:N]$ are in S.

The word "almost all" means "all but a finite number of."

I. The Reciprocity Law for Two Dimensional Local Rings.

The following notations will be used in this chapter.

A: a two dimensional complete local domain which is normal and has finite residue field,

K: the quotient field of A,

P: the set of all prime ideals of height 1 in A, for $\mathfrak{p}\in P$,

$\kappa(\mathfrak{p})$: the residue field of A at $\mathfrak{p}$,

$K_{\mathfrak{p}}$: the completion of K at $\mathfrak{p}$.

1. *The Prime-to-*ch(K) *Case*

First we notice that for each $\mathfrak{p}\in P$, $K_{\mathfrak{p}}$ is a 2-dimensional local field (cf. Notations). For such a field, Kato [12] defined a canonical isomorphism, for any integer m prime to $\mathrm{ch}(K)$,

$$\eta_{\mathfrak{p}}\colon H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2})\simeq \mathbf{Z}/m\mathbf{Z}.$$

Let $r_{\mathfrak{p}}\colon H^3(K, \mu_m^{\otimes 2})\to H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2})$ be the restriction map.

THEOREM 1.1. *For each $a\in H^3(K, \mu_m^{\otimes 2})$, we have $r_{\mathfrak{p}}(a)=0$ for almost all $\mathfrak{p}\in P$, and $\sum_{\mathfrak{p}\in P}\eta_{\mathfrak{p}}\circ r_{\mathfrak{p}}(a)=0$.*

Proof. First we prove Theorem 1.1 in case $A=\mathcal{O}_k[[T]]$, where k is a local field (cf. Notations), $\mathcal{O}_k$ its ring of the integers, and T is a variable. Let A^0 be the henselization of $\mathcal{O}_k[T]$ at the maximal ideal (m_k, T), K^0 its quotient field, $X=\operatorname{Spec} A-\{m\}$, $X^0=\operatorname{Spec} A^0-\{m^0\}$, where m (resp. m^0) is the maximal ideal of A (resp. A^0) and $j\colon X\to X^0$ the natural morphism. Then we have

$$X=\varprojlim_R (X^0\times_{\operatorname{Spec} A^0}\operatorname{Spec} R),$$

where R runs over all subrings of A which are finitely generated over A^0. By Artin [1], for such a ring R, there exists an A^0-homomorphism $R \to A^0$. Hence the homomorphism

$$j^*: H^4(X^0, \mu_m^{\otimes 2}) \to H^4(X, \mu_m^{\otimes 2})$$

is injective. Now we consider the following commutative diagram

$$\begin{tikzcd}
H^3(K^0, \mu_m^{\otimes 2}) \arrow[r, "f"] \arrow[d] & \bigoplus_{\mathfrak{q}_0} H^4_{\mathfrak{q}_0}(X^0, \mu_m^{\otimes 2}) \arrow[r] \arrow[d, "\wr"] & H^4(X^0, \mu_m^{\otimes 2}) \arrow[d, hook] \\
H^3(K, \mu_m^{\otimes 2}) \arrow[r, "g"] & \bigoplus_{\mathfrak{p}} H^4_{\mathfrak{p}}(X, \mu_m^{\otimes 2}) \arrow[r] & H^4(X, \mu_m^{\otimes 2})
\end{tikzcd}$$

where the horizontal sequences are the localization sequences of etale cohomology theory of X and X^0, and $\mathfrak{p}$ (resp. $\mathfrak{q}_0$) runs over all *closed points of X (resp. X^0)*, that is, all prime ideals of height 1 in A (resp. A^0). The middle vertical arrow is an isomorphism, for the prime ideals of A and A^0 are in a one-to-one correspondence, and we have isomorphisms (Kato [12])

$$H^4_{\mathfrak{q}_0}(X^0, \mu_m^{\otimes 2}) \simeq H^3(K^0_{\mathfrak{q}_0}, \mu_m^{\otimes 2}) \simeq \mathbf{Z}/m\mathbf{Z},$$
$$H^4_{\mathfrak{p}}(X, \mu_m^{\otimes 2}) \simeq H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2}) \simeq \mathbf{Z}/m\mathbf{Z},$$

where $K^0_{\mathfrak{q}_0}$ is the completion of K^0 at prime ideal $\mathfrak{q}_0$ of A^0. Consequently we have an isomorphism Image$(f) \simeq$ Image(g). On the other hand, by Kato [13, Sect. 1.8], we know that the composite homomorphism

$$H^3(K^0, \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{q}_0} H^3(K^0_{\mathfrak{q}_0}, \mu_m^{\otimes 2}) \simeq \bigoplus_{\mathfrak{q}_0} \mathbf{Z}/m\mathbf{Z} \xrightarrow{\text{addition}} \mathbf{Z}/m\mathbf{Z}$$

is the zero map. Hence the proof of the special case is completed.

Next we prove Theorem 1.1 in the general case. By Nagata [1, (31.6)], there exists a subring B of A over which A is finite, and which is isomorphic to the ring we treated in the first step of the proof. Let L be the quotient field of B. We have the following commutative diagram

$$\begin{tikzcd}
H^3(K, \mu_m^{\otimes 2}) \arrow[r] \arrow[d, "\mathrm{Cor}_{K/L}"] & \bigoplus_{\mathfrak{q}} \bigoplus_{\mathfrak{p}|\mathfrak{q}} H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2}) \arrow[d, "\oplus_{\mathfrak{p}|\mathfrak{q}} \mathrm{Cor}_{K_{\mathfrak{p}}/L_{\mathfrak{q}}}"] \\
H^3(L, \mu_m^{\otimes 2}) \arrow[r] & \bigoplus_{\mathfrak{q}} H^3(L_{\mathfrak{q}}, \mu_m^{\otimes 2})
\end{tikzcd}$$

where $\mathfrak{q}$ (resp. $\mathfrak{p}$) runs over all prime ideals of height 1 in B (resp. A), $\mathfrak{q} \mid \mathfrak{p}$ means that $\mathfrak{q}$ lies over $\mathfrak{p}$, and $\mathrm{Cor}_{K/L}$ and $\mathrm{Cor}_{K_{\mathfrak{p}}/L_{\mathfrak{q}}}$ are the corestriction maps in Galois cohomology theory. Furthermore we have the following commutative diagram (Kato [12])

$$\begin{tikzcd} H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2}) \simeq \mathbf{Z}/m\mathbf{Z} \arrow[d, "\mathrm{Cor}_{K_{\mathfrak{p}}/L_{\mathfrak{q}}}"] & \\ H^3(L_{\mathfrak{q}}, \mu_m^{\otimes 2}) \simeq \mathbf{Z}/m\mathbf{Z}. & \end{tikzcd}$$

Consequently the general case follows from the special case.

2. *The p-Primary Case*

In this section we suppose that $p = \mathrm{ch}(K)$ is positive, and investigate the p-primary version of Theorem 1.1.

For a field L of characteristic p, let $P_n^r(L)$ be the cokernel of $\mathrm{F}-1\colon \mathrm{W}_n\Omega_L^r \to \mathrm{W}_n\Omega_L^r/d\mathrm{W}_n\Omega_L^{r-1}$, where $\mathrm{W}_n\Omega_L^{\cdot}$ denotes the De Rham–Witt complex on L, and F is the Frobenius map (cf. Illusie [11]). On the other hand, in the higher local class field theory (cf. Kato [13]), Kato defined, for each $\mathfrak{p} \in P$, a canonical isomorphism

$$\mathrm{Res}_{K_{\mathfrak{p}}}^n\colon P_n^2(K_{\mathfrak{p}}) \simeq \mathbf{Z}/p^n\mathbf{Z}.$$

Let $r_{\mathfrak{p}}\colon P_n^2(K) \to P_n^2(K_{\mathfrak{p}})$ be the natural homomorphism. Then the p-primary part of the reciprocity law is the following

THEOREM 2.1. *For each $x \in P_n^2(K)$, $r_{\mathfrak{p}}(x) = 0$ for almost all $\mathfrak{p} \in P$, and we have*

$$\sum_{\mathfrak{p} \in P} \mathrm{Res}_{K_{\mathfrak{p}}}^n \cdot r_{\mathfrak{p}}(x) = 0.$$

Proof. By the definition of P_n^r and the computation of the De Rham–Witt complex (cf. Illusie [11] and Bloch [2]), $P_n^2(K)$ is generated by the classes x of the elements

$$a \cdot d\log r_1 \cdot d\log r_2 \qquad \text{with} \quad a \in \mathrm{W}_n(K) \text{ and } r_1, r_2 \in K^{\times}.$$

On the other hand, by the calculation of the residue map (Kato [3]),

$$\mathrm{Res}_{K_{\mathfrak{p}}}^n(x) = 0 \qquad \text{if} \quad a \in \mathrm{W}_n(R_{\mathfrak{p}}) \text{ and } r_1, r_2 \in R_{\mathfrak{p}}^{\times}, \tag{2.2}$$

where $R_{\mathfrak{p}}$ is the ring of the integers of $K_{\mathfrak{p}}$, and

$$\mathrm{Res}_{K_{\mathfrak{p}}}^n(x) = d_{\mathfrak{p}} \cdot \partial_{\mathfrak{p}}(\{r_1, r_2\}) \cdot \mathrm{Tr}(a) \qquad \text{if} \quad a \in \mathrm{W}_n(\mathbb{F}_q). \tag{2.3}$$

Here, $\mathbb{F}_q \subsetneq A$ is the algebraic closure of $\mathbb{F}_p$ in A, $\partial_{\mathfrak{p}}\colon K_2(K) \to \kappa(\mathfrak{p})^{\times}$ is the boundary map in K-theory, $d_{\mathfrak{p}}\colon \kappa(\mathfrak{p})^{\times} \to \mathbf{Z}$ is the composite homomorphism

$$\kappa(\mathfrak{p})^{\times} \xrightarrow{v_{\mathfrak{p}}} \mathbf{Z} \xrightarrow{n_{\mathfrak{p}}} \mathbf{Z},$$

where $v_{\mathfrak{p}}$ is the normalized additive valuation of the local field $\kappa(\mathfrak{p})$, $n_{\mathfrak{p}}$ is the degree of the residue field of $\kappa(\mathfrak{p})$ over $\mathbb{F}_q$ and $\mathrm{Tr}\colon \mathrm{W}_n(\mathbb{F}_q) \to \mathrm{W}_n(\mathbb{F}_p)$ $(\simeq \mathbf{Z}/p^n\mathbf{Z})$ is the trace map. Thus, the first assertion follows at once from (2.2). For the second assertion, the proof can be reduced to the case $A = \mathbb{F}_q[[T, S]]$ by the same argument as the last part of the proof of Theorem 1.1. Except that we use the trace map $\mathrm{Tr}_{L/K}$ in K-theory (cf. Kato [13, Sect. 2.1, Definition 1]) instead of the corestriction map $\mathrm{Cor}_{L/K}$ in Galois cohomology theory. Now, if $x \in P_n^2(K)$ is (the class of) the element of the form

$$a \cdot d\log r_1 \cdot d\log r_2 \qquad \text{with} \quad a \in \mathrm{W}_n(\mathbb{F}_q) \text{ and } r_1, r_2 \in K^{\times}, \tag{2.4}$$

then by (2.3), our assertion is reduced to the fact that $\sum_{\mathfrak{p} \in P} d_{\mathfrak{p}} \cdot \partial_{\mathfrak{p}} = 0$ which follows from the localization sequence in K-theory (cf. Quillen [20, Sect. 5]. Moreover, we have the following commutative diagram

$$\begin{array}{ccc}
P_n^2(K_{\mathfrak{p}}) & \xrightarrow{\mathrm{Res}^n_{K_{\mathfrak{p}}}} & \mathbf{Z}/p^n\mathbf{Z} = \dfrac{1}{p^n}\mathbf{Z}/\mathbf{Z} \\
\downarrow & & \cap\!\downarrow \\
P_{n+1}^2(K_{\mathfrak{p}}) & \xrightarrow{\mathrm{Res}^{n+1}_{K_{\mathfrak{p}}}} & \mathbf{Z}/p^{n+1}\mathbf{Z} = \dfrac{1}{p^{n+1}}\mathbf{Z}/\mathbf{Z},
\end{array}$$

where the left vertical map is induced by

$$\mathrm{V}\colon \mathrm{W}_n\Omega^2_{K_{\mathfrak{p}}} \to \mathrm{W}_{n+1}\Omega^2_{K_{\mathfrak{p}}}.$$

Therefore the following lemma suffices to prove Theorem 2.1.

LEMMA 2.5. *Let* $A = \mathbb{F}_q[[T, S]]$ *and* D_n *the subgroup of* $P_n^2(K)$ *generated by all elements of the form* (2.4). *Then we have*

$$\varinjlim_n D_n = \varinjlim_n P_n^2(K).$$

For the proof of Lemma 2.5, we need

LEMMA 2.6. *Let A be as above and Q be the set of all prime ideals of height* 1 *in A but the prime ideal* (S). *Then we have the isomorphism* $\bigoplus_{\mathfrak{p} \in Q} \mathrm{Res}^n_{K_{\mathfrak{p}}} \cdot r_{\mathfrak{p}}\colon P_n^2(K) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \in Q} \mathbf{Z}/p^n\mathbf{Z}.$

Assuming Lemma 2.6, we prove Lemma 2.5. Put $C_n = H^1(\mathbb{F}_q, \mathbf{Z}/p^n\mathbf{Z})$. By the theory of Artin–Schreier–Witt, we have the isomorphism

$$s_n\colon C_n \xrightarrow{\sim} \mathrm{W}_n(\mathbb{F}_q)/(\mathrm{F}-1)\mathrm{W}_n(\mathbb{F}_q).$$

Then we define the homomorphism

$$u_n\colon C_n \otimes K_2(K) \to P_n^2(K);\ a \otimes \{r_1, r_2\} \mapsto s_n(a) \cdot d\log r_1 \cdot d\log r_2.$$

By definition, we have the commutative diagram

$$\begin{tikzcd}
C_n \otimes K_2(K) \arrow[r, "u_n"] \arrow[d] & P_n^2(K) \arrow[d] \\
C_{n+1} \otimes K_2(K) \arrow[r, "u_{n+1}"] & P_{n+1}^2(K),
\end{tikzcd}$$

and the assertion of Lemma 2.5 is equivalent to the surjectivity of the map $\varinjlim_n u_n$. On the other hand, we have the isomorphism

$$t_n\colon C_n \simeq \mathbf{Z}/p^n\mathbf{Z};\ \chi \mapsto \chi \text{ (the Frobenius over } \mathbb{F}_q),$$

and we define, for each $\mathfrak{p} \in P$, the homomorphism

$$v_\mathfrak{p}^n\colon C_n \otimes \kappa(\mathfrak{p})^\times \to \mathbf{Z}/p^n\mathbf{Z};\ \chi \otimes r \mapsto d_\mathfrak{p}(r) \cdot t_n(\chi).$$

Then the following diagram is commutative:

$$\begin{tikzcd}
C_n \otimes K_2(K) \arrow[r, two heads, "\oplus \mathrm{id} \otimes \partial_\mathfrak{p}"] \arrow[d, "u_n"] & \bigoplus_{\mathfrak{p} \in Q} C_n \otimes \kappa(\mathfrak{p})^\times \arrow[d, "\oplus v_\mathfrak{p}^n"] \\
P_n^2(K) \arrow[r, "(2.6)", "\sim"', "\oplus \mathrm{Res}_{K_\mathfrak{p}}^n \cdot r_\mathfrak{p}"'] & \bigoplus_{\mathfrak{p} \in Q} \mathbf{Z}|p^n\mathbf{Z}.
\end{tikzcd}$$

The surjectivity of the upper horizontal arrow follows at once from Quillen [20, Theorem 5.13]. Consequently the surjectivity of the map $\varinjlim_n u_n$ follows from that of the map $\varinjlim_n v_\mathfrak{p}^n$, and it can be easily verified.

Now we prove (2.6). Let $B = A[1/S]$, $X = \operatorname{Spec} B$ and v_n be the sheaf $\mathrm{W}_n\Omega^2_{X,\log}$ on $X_{ét}$ ("the logarithmic part of $\mathrm{W}_n\Omega^2_X$" cf. Illusie [11, 5.7]). We have the localization sequence on X

$$H^1(X, v_n) \to H^1(K, v_n) \to \bigoplus_{\mathfrak{p} \in Q} H^2_\mathfrak{p}(X, v_n) \to H^2(X, v_n). \tag{2.7}$$

On the other hand, in view of the isomorphism $H_\mathfrak{p}^*(X, v_n) \simeq H_\mathfrak{p}^*(X_\mathfrak{p}^h, v_n)$ ($X_\mathfrak{p}^h$ is the henselization of X at $\mathfrak{p}$), the localization sequence on $X_\mathfrak{p}^h$ and an easy computation give us the isomorphism $H_\mathfrak{p}^2(X, v_n) \simeq H^1(K_\mathfrak{p}^h, v_n)$, where $K_\mathfrak{p}^h$ denotes the henselization of K at $\mathfrak{p}$. Furthermore we have $H^2(X, v_n) = 0$ and $H^1(X, v_n) = 0$. In fact, the first assertion follows from S.G.A.4X

Theorem 5.1, for X is affine, so $cdqcX = 1$. Using the exact sequence (cf. [4, Sect. 1.4, Lemma 3]),

$$0 \to \nu_n \xrightarrow{p^m} \nu_{n+m} \xrightarrow{R^n} \nu_m \to 0 \ (m, n;\ \text{integer} > 0),$$

the proof of the second assertion is reduced to the case $n = 1$: By the exact sequence $0 \to \nu_1 \to \Omega_X^2 \to^{1-\gamma} \Omega_X^2 \to 0$ (γ is the Cartier operator), we have an isomorphism

$$\Omega_B^2/(1-\gamma)\Omega_B^2 \simeq H^1(X, \nu_1),$$

and our assertion follows by an easy computation.

Now, by the exact sequence (2.7), we have an isomorphism

$$H^1(K, \nu_n) \simeq \bigoplus_{\mathfrak{p} \in Q} H^1(K_{\mathfrak{p}}^h, \nu_n).$$

Noting that $\mathrm{Res}_{K_{\mathfrak{p}}}^n$ induces an isomorphism $P_n^2(K_{\mathfrak{p}}^h) \simeq \mathbf{Z}/p^n\mathbf{Z}$, the following lemma is sufficient to complete the proof.

LEMMA 2.8. *Let L be a field of characteristic $p > 0$. Then there exists a canonical isomorphism*

$$P_n^2(L) \simeq H^1(L, \nu_n),$$

where ν_n is the sheaf $\mathrm{W}_n\Omega_{L,\log}^2$ *on* $(\mathrm{Spec}\ L)_{\acute{e}t}$

Proof. By the exact sequence (cf. [4, Sect. 1.4, Lemma 2]),

$$0 \to \nu_n \to \mathrm{W}_n\Omega_L^2 \xrightarrow{1-\mathrm{F}} \mathrm{W}_n\Omega_L^2/d\mathrm{V}^{n-1}\Omega_L^1 \to 0,$$

we have an isomorphism

$$\mathrm{Coker}\ (1-\mathrm{F}\colon \mathrm{W}_n\Omega_L^2 \to \mathrm{W}_n\Omega_L^2/d\mathrm{V}^{n-1}\Omega_L^1) \simeq H^1(L, \nu_n).$$

Hence it suffices to show

$$(1-\mathrm{F})(\mathrm{W}_n\Omega_L^2 + d\mathrm{V}^{n-1}\Omega_L^1 = (1-\mathrm{F})\,\mathrm{W}_n\Omega_L^2 + d\mathrm{W}_n\Omega_L^1,$$

and it follows by an easy computation on the De Rahm–Witt complex and left to the readers.

3. *Some Consequences of the Reciprocity Law*

First we recall the class field theory of $K_{\mathfrak{p}}$ established by Kato [12, 13]. He defined a canonical pairing

$$\langle\ \rangle_{\mathfrak{p}}\colon H^1(K_{\mathfrak{p}}, \mathbf{Z}/m\mathbf{Z}) \otimes K_2(K_{\mathfrak{p}}) \to \mathbf{Z}/m\mathbf{Z} \qquad (m > 0)$$

as follows: Put $p=\mathrm{ch}(K)\geqslant 0$. In case m is prime to p, using the isomorphism $\eta_{\mathfrak{p}}: H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2})\simeq \mathbf{Z}/m\mathbf{Z}$ (cf. Sect. 1), we define, for $\chi\in H^1(K_{\mathfrak{p}}, \mathbf{Z}/m\mathbf{Z})$ and $a\in K_2(K_{\mathfrak{p}})$,

$$\langle \chi, a\rangle_{\mathfrak{p}} = \eta_{\mathfrak{p}}(\chi \cup h_{\mathfrak{p}}^2(a)),$$

where $h_{\mathfrak{p}}^2: K_2(K_{\mathfrak{p}})\to H^2(K_{\mathfrak{p}}, \mu_m^{\otimes 2})$ is the Galois symbol on $K_{\mathfrak{p}}$ (cf. Tate [26]), and $\cup$ denotes the cup product in Galois cohomology theory. Next we assume that p is positive and $m=p^n$ $(n>0)$. Let $\mathrm{W}_n(K_{\mathfrak{p}})$ be the group of all p-Witt vectors of length n, and $\mathrm{F}: \mathrm{W}_n(K_{\mathfrak{p}})\to \mathrm{W}_n(K_{\mathfrak{p}})$ be the Frobenius map. By the definition of $P_n^2(K_{\mathfrak{p}})$ (cf. Sect. 2), we have a canonical pairing

$$\mathrm{W}_n(K_{\mathfrak{p}})/(1-\mathrm{F})\,\mathrm{W}_n(K_{\mathfrak{p}})\otimes K_2(K_{\mathfrak{p}})\to P_n^2(K_{\mathfrak{p}}).$$

On the other hand, we have $\mathrm{Res}_{K_{\mathfrak{p}}}^n: P_n^2(K_{\mathfrak{p}})\simeq \mathbf{Z}/p^n\mathbf{Z}$ (cf. Sect. 2), and the isomorphism $\mathrm{W}_n(K_{\mathfrak{p}})/(1-\mathrm{F})\,\mathrm{W}_n(K_{\mathfrak{p}})\simeq H^1(K_{\mathfrak{p}}, \mathbf{Z}/p^n\mathbf{Z})$ from the theory of Artin–Schreier–Witt. Hence we obtain the desired pairing.

Now, the class field theory of $K_{\mathfrak{p}}$ is stated as follows.

THEOREM 3.1. (1) *The pairing* $\langle\ \rangle_{\mathfrak{p}}$ *induces an isomorphism*

$$\Phi_{\mathfrak{p}}: H^1(K_{\mathfrak{p}}, \mathbb{Q}/\mathbf{Z})\simeq \mathrm{Hom}_{\mathrm{cf}}(K_2(K_{\mathfrak{p}}), \mathbb{Q}/\mathbf{Z}),$$

where $K_2(K_{\mathfrak{p}})$ *is endowed with the topology defined in Kato* [12, 13], *and* $\mathrm{Hom}_{\mathrm{cf}}$ *denotes the group of continuous homomorphisms of finite order.*

(2) *We have the following commutative diagram*

$$\begin{array}{ccc} K_2(K_{\mathfrak{p}}) & \longrightarrow & \mathrm{Gal}(K_{\mathfrak{p}}^{\mathrm{ab}}/K_{\mathfrak{p}}) \\ {\scriptstyle \partial_{\mathfrak{p}}}\downarrow & & \downarrow \\ \kappa(\mathfrak{p})^\times & \longrightarrow & \mathrm{Gal}(\kappa(\mathfrak{p})^{\mathrm{ab}}/\kappa(\mathfrak{p})) \end{array}$$

where the horizontal arrows come from the class field theory of $K_{\mathfrak{p}}$ *and* $\kappa(\mathfrak{p})$, *and the left vertical arrow is the boundary homomorphism in K-theory. In particular, an element* $\chi\in H^1(K_{\mathfrak{p}}, \mathbb{Q}/\mathbf{Z})$ *is unramified, that is, the corresponding cyclic extension of* $K_{\mathfrak{p}}$ *is unramified if and only if* $\Phi_{\mathfrak{p}}(\chi)$ *is trivial on* $\mathrm{Ker}(\partial_{\mathfrak{p}})$. (*We know that* $\mathrm{Ker}(\partial_{\mathfrak{p}})$ *is the closed subgroup* $K_2(R_{\mathfrak{p}})$, *where* $R_{\mathfrak{p}}$ *denotes the ring of the integers of* $K_{\mathfrak{p}}$.)

Next we "globalize" the pairings $\langle\ \rangle_{\mathfrak{p}}$ for $\mathfrak{p}\in P$. Let I_K be the restricted product $\prod'_{\mathfrak{p}\in P} K_2(K_{\mathfrak{p}})$ with respect to the subgroups $K_2(R_{\mathfrak{p}})$. By definition

I_K consists of all elements $(a_\mathfrak{p})_{\mathfrak{p}\in P}$ such that for almost all $\mathfrak{p}\in P$, $a_\mathfrak{p}$ is contained in $K_2(R_\mathfrak{p})$. For a finite subset S of P, put

$$J^S = \prod_{\mathfrak{p}\in S} K_2(K_\mathfrak{p}) \times \prod_{\mathfrak{p}\in P-S} K_2(R_\mathfrak{p}).$$

Then J^S is a subgroup of I_K and we have $I_K = \varinjlim_S J^S$, where S runs over all finite subsets of P. We endow J^S with the product topology, and make I_K a topological group as the inductive limit of the topological groups J^S.

Let χ be an element of $H^1(K, \mathbb{Q}/\mathbf{Z})$, and $\chi_\mathfrak{p}$ be the restriction of χ to $H^1(K_\mathfrak{p}, \mathbb{Q}/\mathbf{Z})$. Note that $\chi_\mathfrak{p}$ is unramified for almost all $\mathfrak{p}\in P$. Hence if $a=(a_\mathfrak{p})_{\mathfrak{p}\in P}$ is an element of I_K, by Theorem 3.1(2) and the definition of I_K, we have $\langle \chi_\mathfrak{p}, a_\mathfrak{p}\rangle_\mathfrak{p} = 0$ for almost all $\mathfrak{p}\in P$. Consequently we obtain a pairing

$$\langle\ \rangle_K\colon H^1(K, \mathbb{Q}/\mathbf{Z}) \otimes I_K \to \mathbb{Q}/\mathbf{Z},$$

defined by $\langle \chi, (a_\mathfrak{p})_{\mathfrak{p}\in P}\rangle_K = \sum_{\mathfrak{p}\in P} \langle \chi_\mathfrak{p}, a_\mathfrak{p}\rangle_\mathfrak{p}$, and we obtain the induced homomorphism

$$\tilde{\Phi}_K\colon H^1(K, \mathbb{Q}/\mathbf{Z}) \to \mathrm{Hom}_{\mathrm{cf}}(I_K, \mathbb{Q}/\mathbf{Z})$$

On the other hand, the reciprocity law (Theorems 1.1 and 2.1) tells us that if $a=(a_\mathfrak{p})_{\mathfrak{p}\in P}$ is in the diagonal image of $K_2(K)$ in I_K, we have, for any $\chi\in H^1(K, \mathbb{Q}/\mathbf{Z})$,

$$\langle \chi, a\rangle_K = 0;$$

in other words, the homomorphism $\tilde{\Phi}_K$ is trivial on the diagonal image of $K_2(K)$ in I_K. Consequently, denoting by C_K the quotient of I_K by the image of $K_2(K)$, we obtain the homomorphism

$$\Phi_K\colon H^1(K, \mathbb{Q}/\mathbf{Z}) \to \mathrm{Hom}_{\mathrm{cf}}(C_K, \mathbb{Q}/\mathbf{Z}).$$

PROPOSITION 3.2. *If A is regular, Φ_K is injective.*

Proof. Let χ be an element of $H^1(K, \mathbb{Q}/\mathbf{Z})$ and suppose $\Phi_K(\chi)=0$. Let L be the cyclic extension of K corresponding to $\mathrm{Ker}(\chi)$. By the assumption, if we put $\chi_\mathfrak{p}$ the restriction of χ to $H^1(K_\mathfrak{p}, \mathbb{Q}/\mathbf{Z})$, then $\Phi_\mathfrak{p}(\chi_\mathfrak{p})$ ($\in \mathrm{Hom}_{\mathrm{cf}}(K_2(K_\mathfrak{p}), \mathbb{Q}/\mathbf{Z})$) is trivial for any $\mathfrak{p}\in P$. By the class field theory of $K_\mathfrak{p}$ (Theorem 3.1), this implies that any $\mathfrak{p}\in P$ splits completely in the extension L/K; in other words, for any $\mathfrak{p}\in P$, $K_\mathfrak{p}\otimes_K L$ is isomorphic to the direct product of finite copies of $K_\mathfrak{p}$. By the regularity of A, we can conclude that the extension L/K is trivial, that is, $\chi=0$.

PROPOSITION 3.3. *Let L be a finite abelian extension of K in which almost all $\mathfrak{p}\in P$ split completely. Then all $\mathfrak{p}\in P$ split completely in the extension. In particular, if A is regular, the extension is trivial.*

Proof. We may suppose that L is a cyclic extension of K. Let χ be the element of $H^1(K, \mathbb{Q}/\mathbf{Z})$ corresponding to L, and put $\tilde{\chi} = \tilde{\Phi}_K(\chi) \in \mathrm{Hom}_{\mathrm{cf}}(I_K, \mathbb{Q}/\mathbf{Z})$. By the assumption, there exists a finite subset S of P such that $\tilde{\chi}$ is trivial on the subgroup $\prod'_{\mathfrak{p} \in P-S} K_2(K_\mathfrak{p})$ of I_K. On the other hand, by the definition of the topology of $K_2(K_\mathfrak{p})$ and a well-known approximation theorem for a Dedeking domain, we can see that the image of

$$K_2(K) \to \prod_{\mathfrak{p} \in S} K_2(K_\mathfrak{p})$$

is dense. Consequently $\tilde{\chi}$ is trivial on I_K, and by definition, this implies that all $\mathfrak{p} \in P$ split completely in the extension. The last assertion follows from Proposition 3.2.

PROPOSITION 3.4. *Let S be a finite subset of P and A_S be the subring of K consisting of the elements which are integral at any $\mathfrak{p} \in P - S$. Let χ be an element of $H^1(\mathrm{Spec}\, A_S, \mathbb{Q}/\mathbf{Z})$, $\chi_\mathfrak{p}$ be the restriction of χ to $H^1(K_\mathfrak{p}, \mathbb{Q}/\mathbf{Z})$, and $\tilde{\chi}_\mathfrak{p} = \Phi_\mathfrak{p}(\chi_\mathfrak{p}) \in \mathrm{Hom}_{\mathrm{cf}}(K_2(K_\mathfrak{p}), \mathbb{Q}/\mathbf{Z})$. Let $j_\mathfrak{p}$ be the composite homomorphism*

$$K_2(A_S) \to K_2(K) \to K_2(K_\mathfrak{p}).$$

Then for any $a \in K_2(A_S)$, we have

$$\sum_{\mathfrak{p} \in S} \tilde{\chi}_\mathfrak{p}(j_\mathfrak{p}(a)) = 0.$$

Proof. This follows from the reciprocity law (Theorems 1.1 and 2.1), using the fact that if $\mathfrak{p} \in P - S$ the image of $K_2(A_S)$ under the map $j_\mathfrak{p}$ is contained in $K_2(R_\mathfrak{p})$ which $\tilde{\chi}_\mathfrak{p}$ annihilates, for χ is unramified at any $\mathfrak{p} \in P - S$ (cf. Theorem 3.1(2)).

II. CURVES OVER LOCAL FIELDS

The following notations will be used in this chapter.

- k: a local field (cf. Notations),
- X: a projective smooth geometrically irreducible curve over k,
- K: the function field of X,
- P: the set of all closed points of X,

for $\mathfrak{p} \in P$,

- $\kappa(\mathfrak{p})$: the residue field of $\mathfrak{p}$,
- $K_\mathfrak{p}$: the completion of K at $\mathfrak{p}$,
- $\mathrm{ord}_\mathfrak{p}$: the normalized additive valuation of K defined by $\mathfrak{p}$.

1. *The Construction of the Map* $\sigma\colon SK_1(X) \to \pi_1^{ab}(X)$

First, following Bloch [3], we introduce the group $SK_1(X)$ which is a fundamental tool to describe $\pi_1^{ab}(X)$.

DEFINITION 1.1. $SK_1(X)$ is defined to be the cokernel of

$$\bigoplus \partial_{\mathfrak{p}}\colon K_2(K) \to \bigoplus_{\mathfrak{p} \in P} \kappa(\mathfrak{p})^\times,$$

where $\partial_{\mathfrak{p}}\colon K_2(K) \to \kappa(\mathfrak{p})^\times$ is the boundary map in K-theory, which is defined, for $\{f, g\} \in K_2(K)$, as follows:

$$\partial_{\mathfrak{p}}\{f, g\} = (-1)^{\mathrm{ord}_{\mathfrak{p}}(f)\mathrm{ord}_{\mathfrak{p}}(g)} f^{\mathrm{ord}_{\mathfrak{p}}(g)} g^{-\mathrm{ord}_{\mathfrak{p}}(f)}|_{\mathfrak{p}}.$$

In this section, we construct a fundamental homomorphism

$$\sigma\colon SK_1(X) \to \pi_1^{ab}(X),$$

which is the starting point of the class field theory of X. First, notice that for $\mathfrak{p} \in P$, $K_{\mathfrak{p}}$ is a 2-dimensional local field (cf. Notations). As explained in Chapter I, Section 3, there is a canonical pairing

$$\langle\ \rangle_{\mathfrak{p}}\colon H^1(K_{\mathfrak{p}}, \mathbb{Q}/\mathbf{Z}) \otimes K_2(K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbf{Z},$$

and these pairing for $\mathfrak{p} \in P$ are "globalized" to the pairing

$$\langle\ \rangle_K\colon H^1(K, \mathbb{Q}/\mathbf{Z}) \otimes I_K \to \mathbb{Q}/\mathbf{Z}.$$

Here I_K denotes the restricted product $\prod'_{\mathfrak{p} \in P} K_2(K_{\mathfrak{p}})$ with respect to the subgroups $K_2(R_{\mathfrak{p}})$ (cf. Chap. I, Sect. 3). As in Chapter I, we have the following

PROPOSITION 1.2. *If $a = (a_{\mathfrak{p}})_{\mathfrak{p} \in P}$ is in the diagonal image of $K_2(K)$ in I_K, we have, for any $\chi \in H^1(K, \mathbb{Q}/\mathbf{Z})$,*

$$\langle \chi, a \rangle_K = 0.$$

Proof. We use the same notation as in Chapter I. As explained in Chapter I, Section 3, it suffices to show the same assertion as Theorems 1.1 and 2.1 in Chapter I for the present context. But the proof is much easier. First we prove the first assertion. Consider the localization sequence in etale cohomology theory of X,

$$H^3(X, \mu_m^{\otimes 2}) \to H^3(K, \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{p} \in P} H^4_{\mathfrak{p}}(X, \mu_m^{\otimes 2}) \to H^4(X, \mu_m^{\otimes 2}). \qquad (1.3)$$

As explained in Chapter I, Section 1, we have an isomorphism

$$t_{\mathfrak{p}}\colon H^4_{\mathfrak{p}}(X, \mu_m^{\otimes 2}) \xrightarrow{\sim} H^3(K_{\mathfrak{p}}, \mu_m^{\otimes 2}) \xrightarrow{\sim} \mathbf{Z}/m\mathbf{Z},$$

through which the homomorphism $H^3(K, \mu_m^{\otimes 2}) \to H^4_{\mathfrak{p}}(X, \mu_m^{\otimes 2})$ is identified with the homomorphism $\eta_{\mathfrak{p}} \circ r_{\mathfrak{p}}$. On the other hand, we have a Hochshild–Serre spectral sequence

$$H^p(k, H^q(\bar{X}, \mu_m^{\otimes 2})) \Rightarrow H^{p+q}(X, \mu_m^{\otimes 2}),$$

where $\bar{X} = X \times_k k^{\mathrm{sep}}$. From this, we can deduce an isomorphism

$$t\colon H^4(X, \mu_m^{\otimes 2}) \xrightarrow{\sim} H^2(k, H^2(\bar{X}, \mu_m^{\otimes 2})) \xrightarrow{\sim} H^2(k, \mu_m) \xrightarrow{\sim} \mathbf{Z}/m\mathbf{Z}.$$

Furthermore the following diagram is commutative,

$$\begin{tikzcd} H^4_{\mathfrak{p}}(X, \mu_m^{\otimes 2}) \arrow[r] \arrow[d, "\wr\, t_{\mathfrak{p}}"] & H^4(X, \mu_m^{\otimes 2}) \arrow[d, "\wr\, t"] \\ \mathbf{Z}/m\mathbf{Z} \arrow[r, equal] & \mathbf{Z}/m\mathbf{Z}. \end{tikzcd}$$

Consequently the first assertion follows from the sequence (1.3).

Next we treat the second assertion. By the computation of the residue map in Milnor K-theory in Kato [14], our assertion follows from Kato [14, Proposition 4, Sect. 3].

By Proposition 1.2 we have a canonical pairing

$$\langle\ \rangle_K\colon H^1(K, \mathbb{Q}/\mathbf{Z}) \otimes C_K \to \mathbb{Q}/\mathbf{Z},$$

where C_K is the quotient of I_K by the diagonal image of $K_2(K)$. On the other hand, if χ is an element of $H^1(X, \mathbb{Q}/\mathbf{Z})$, then the image of χ in $H^1(K_{\mathfrak{p}}, \mathbb{Q}/\mathbf{Z})$ is unramified for any $\mathfrak{p} \in P$, so by Chapter I, Theorem 3.1, for any $a = (a_{\mathfrak{p}})_{\mathfrak{p} \in P}$ which is in the subgroup $\prod_{\mathfrak{p} \in P} K_2(R_{\mathfrak{p}})$, we have $\langle \chi, a \rangle_K = 0$. Now we notice that the quotient of C_K by the image of the subgroup $\prod_{\mathfrak{p} \in P} K_2(R_{\mathfrak{p}})$ is canonically isomorphic to $SK_1(X)$, for the quotient of $K_2(K_{\mathfrak{p}})$ by $K_2(R_{\mathfrak{p}})$ is isomorphic to $\kappa(\mathfrak{p})^\times$ via the boundary map $\partial_{\mathfrak{p}}$. Consequently $\langle\ \rangle_K$ induces the pairing $H^1(X, \mathbb{Q}/\mathbf{Z}) \otimes SK_1(X) \to \mathbb{Q}/\mathbf{Z}$, and we obtain the desired homomorphism $\sigma\colon SK_1(X) \to H^1(X, \mathbb{Q}/\mathbf{Z})^* = \pi_1^{\mathrm{ab}}(X)$.

2. *The Cokernel of σ*

In this section we determine the cokernel of σ. For this we introduce the following notion.

DEFINITION 2.1. Let Z be a noetherian scheme. A finite etale covering $f\colon U \to Z$ is called a *c.s. covering*, if any closed point x of Z *splits completely* in the covering, that is, $\operatorname{Spec} \kappa(x) \times_Z U$ is isomorphic to a finite sum of

Spec $\kappa(x)$, where $\kappa(x)$ is the residue field of x. We denote by $\pi_1^{\text{c.s.}}(Z)$ the quotient group of $\pi_1^{\text{ab}}(Z)$ which classifies *abelian* c.s. coverings of Z.

By definition the quotient of $\pi_1^{\text{ab}}(X)$ by the closure $\overline{\text{Im}(\sigma)}$ of the image of σ classifies abelian c.s. coverings of X, in other words, we have the isomorphism

$$\pi_1^{\text{ab}}(X)/\overline{\text{Im}(\sigma)} \simeq \pi_1^{\text{c.s.}}(X).$$

Now, to investigate the structure of $\pi_1^{\text{c.s.}}(X)$, we fix a *regular model* $\mathscr{X}$ *of* X *over* Spec $\mathcal{O}_k$, that is, a 2-dimensional regular scheme $\mathscr{X}$ with a proper flat morphism $\pi: \mathscr{X} \to \text{Spec}\, \mathcal{O}_k$ such that $\mathscr{X} \times_{\mathcal{O}_k} k \simeq X$. For the existence of such a scheme, we refer to [6]. We denote by $\mathscr{X}_s$ the special fiber $\mathscr{X} \times_{\mathcal{O}_k} F$, where F is the residue field of k.

PROPOSITION 2.2. *There exists a canonical isomorphism*

$$\pi_1^{\text{c.s.}}(X) \xrightarrow{\sim} \pi_1^{\text{c.s.}}(\mathscr{X}_s).$$

Proof. In view of S.G.A.X Theorem 2.1, we have the natural homomorphism $\pi_1^{\text{c.s.}}(X) \to \pi_1^{\text{c.s.}}(\mathscr{X}_s)$, and its surjectivity is trivial. The main point to prove is the following

Claim. Let $f: Y \to X$ be an abelian c.s. covering and $\mathscr{Y}$ the integral closure of $\mathscr{X}$ in the function field L of Y, $\bar{f}: \mathscr{Y} \to \mathscr{X}$ the extension of f. Then $\bar{f}$ is etale, and $f_s: \mathscr{Y}_s \to \mathscr{X}_s$, the restriction of f to the special fibers, is a c.s. covering.

Proof. For a closed point x of $\mathscr{X}_s$, let R_x be the completion of the local ring of $\mathscr{X}$ at x and K_x the quotient field of R_x. For $\mathfrak{p} \in P$, the closure $\overline{\{\mathfrak{p}\}}$ of $\mathfrak{p}$ in $\mathscr{X}$ contains the unique closed point x of $\mathscr{X}_s$ (we denote this relation by $\mathfrak{p} \to x$), and $\overline{\{\mathfrak{p}\}}$ determines a prime ideal of height 1 in R_x. Thus any prime ideal of height 1 in R_x, except those finite number of prime ideals lying over the maximal ideal of $\mathcal{O}_k$, comes from some $\mathfrak{p} \in P$ such that $\mathfrak{p} \to x$. Now, since f is a c.s. covering, in the extension $K_x \otimes_K L$ of K_x, any prime ideal coming from some $\mathfrak{p} \in P$ splits completely. Hence, by Chapter I, Proposition 3.3, the extension is trivial, that is, $R_x \otimes_{\mathscr{X}} \mathscr{Y}$ is isomorphic to a direct product of finite copies of R_x. Clearly this suffices to prove our assertion.

Next we investigate the structure of $\pi_1^{\text{c.s.}}(\mathscr{X}_s)$. Since we are dealing with etale coverings of $\mathscr{X}_s$, we may replace $\mathscr{X}_s$ by its reduced part $(\mathscr{X}_s)_{\text{red}}$. Furthermore, by blowing up some closed points of $\mathscr{X}$ in advance, we can suppose $(\mathscr{X}_s)_{\text{red}}$ has only ordinary double points as singularities. Now we introduce the following notion.

DEFINITION 2.3. The graph Γ of $(\mathscr{X}_s)_{\mathrm{red}}$ is defined by:

(1) The set of vertices of Γ is the set of irreducible components of $(\mathscr{X}_s)_{\mathrm{red}}$.

(2) The set of edges of Γ is the set of singular points of $(\mathscr{X}_s)_{\mathrm{red}}$.

(3) An edge x has for extremities the irreducible components of $(\mathscr{X}_s)_{\mathrm{red}}$ on which x lies.

THEOREM 2.4. *There exists an isomorphism*

$$\pi_1^{\mathrm{c.s.}}(\mathscr{X}_s) \simeq \hat{\mathbf{Z}} \otimes_{\mathbf{Z}} H_1(\Gamma, \mathbf{Z}) \simeq \hat{\mathbf{Z}}^r,$$

where r is the rank of $H_1(\Gamma, Z)$.

DEFINITION 2.5. "The rank of X over k" is the rank of $H_1(\Gamma, \mathbf{Z})$.

We conclude that it does not depend on the choice of $\mathscr{X}$ from the following

THEOREM 2.6. *The quotient of $\pi_1^{\mathrm{ab}}(X)$ by the closure of the image of $\sigma\colon SK_1(X) \to \pi_1^{\mathrm{ab}}(X)$ is isomorphic to $\hat{\mathbf{Z}}^r$, where r is the rank of X over k.*

Theorem 2.6 follows at once from Theorem 2.4 and Proposition 2.2.

Proof of Theorem 2.4. Put $C = (\mathscr{X}_s)_{\mathrm{red}}$ and let U be the open dense subscheme of C consisting of the regular points of C, and put $S = C - U$. Let $H^1(C, \mathbf{Q}/\mathbf{Z})_{\mathrm{c.s.}}$ be the subgroup of $H^1(C, \mathbf{Q}/\mathbf{Z})$ corresponding to abelian c.s. coverings of C. It is the Pontriyagin dual of $\pi_1^{\mathrm{c.s.}}(C)$. Consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \bigoplus_{s \in S} H^0(x, \mathbf{Q}/\mathbf{Z}) \arrow[d] & & \\
 & & \bigoplus_{x \in S} H^0(\mathrm{Spec}\, \mathcal{O}^h_{C,x} - x, \mathbf{Q}/\mathbf{Z}) \arrow[d] & & \\
H^0(C, \mathbf{Q}/\mathbf{Z}) \arrow[r] & H^0(U, \mathbf{Q}/\mathbf{Z}) \arrow[r] & H^1_S(C, \mathbf{Q}/\mathbf{Z}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^1(C, \mathbf{Q}/\mathbf{Z}) \arrow[r, "f"] \arrow[dl, "g"] & H^1(U, \mathbf{Q}/\mathbf{Z}) \\
 & & \bigoplus_{x \in S} H^1(x, \mathbf{Q}/\mathbf{Z}) & &
\end{tikzcd}$$

where $\mathcal{O}^h_{C,x}$ is the henselization of the local ring of C at x. The horizontal and vertical sequences are localization sequences in etale cohomology theory and they are exact. We use the following result from the class field theory for curves over finite field: Let V be a smooth connected curve over a finite field. Then V has no nontrivial c.s. covering. From the result, we can conclude

$$H^1(C, \mathbb{Q}/\mathbf{Z})_{\text{c.s.}} = \operatorname{Ker}(f) \cap \operatorname{Ker}(g).$$

Consequently we have an exact sequence

$$H^0(C, \mathbb{Q}/\mathbf{Z}) \to H^0(U, \mathbb{Q}/\mathbf{Z}) \to \bigoplus_{x \in S} \frac{H^0(\operatorname{Spec} \mathcal{O}^h_{C,x} - x, \mathbb{Q}/\mathbf{Z})}{H^0(x, \mathbb{Q}/\mathbf{Z})}$$
$$\to H^1(C, \mathbb{Q}/\mathbf{Z})_{\text{c.s.}} \to 0.$$

On the other hand, we have isomorphisms

$$H^0(C, \mathbb{Q}/\mathbf{Z}) \simeq \mathbb{Q}/\mathbf{Z} \qquad \text{and} \qquad H^0(U, \mathbb{Q}/\mathbf{Z}) \simeq \bigoplus_{v} \mathbb{Q}/\mathbf{Z},$$

where v runs over all irreducible components of C, and for any $x \in S$, we have an isomorphism

$$H^0(\operatorname{Spec} \mathcal{O}^h_{C,x} - x, \mathbb{Q}/\mathbf{Z}) \simeq \bigoplus_{\eta} \mathbb{Q}/\mathbf{Z} = \mathbb{Q}/\mathbf{Z} \oplus \mathbb{Q}/\mathbf{Z},$$

where η runs over all maximal points of Spec $\mathcal{O}^h_{C,x}$, and the number of those points is two because $x \in S$ is an ordinary double point on C. Hence the cokernel of

$$H^0(x, \mathbb{Q}/\mathbf{Z}) \to H^0(\operatorname{Spec} \mathcal{O}^h_{C,x} - x, \mathbb{Q}/\mathbf{Z})$$

is isomorphic to $\mathbb{Q}/\mathbf{Z}$. Therefore, we have an exact sequence

$$\bigoplus_{v} \mathbb{Q}/\mathbf{Z} \to \bigoplus_{x \in S} \mathbb{Q}/\mathbf{Z} \to H^1(C, \mathbb{Q}/\mathbf{Z})_{\text{c.s.}} \to 0,$$

where by definition each v corresponds to a vertex of Γ, and each x corresponds to an edge of Γ. Consequently, we have

$$H^1(C, \mathbb{Q}/\mathbf{Z})_{\text{c.s.}} \xrightarrow{\sim} H^1(\Gamma, \mathbb{Q}/\mathbf{Z}).$$

Then Theorem 2.4 follows considering the dual groups.

EXAMPLE 2.7. For each integer $m > 0$, let E_m be the Tate curve $\mathbb{G}_m/q^{m\mathbf{Z}}$,

FIG. 1. An m-polygon of smooth rational curves for $m=1$ and $m \geqslant 2$, respectively.

where q is a fixed prime element of k. By definition, there exists a canonical isomorphism

$$E_m(\bar{k}) \simeq \bar{k}^\times / q^{m\mathbf{Z}}.$$

The minimal model of E_m over Spec $\mathcal{O}_k$ has the following special fiber (in particular, the rank of E_m is 1) (see Fig. 1).

Let $\varphi_m: E_m \to E_1$ be the isogeny corresponding to the natural homomorphism $\bar{k}^\times / q^{m\mathbf{Z}} \to \bar{k}^\times / q^{\mathbf{Z}}$. Then φ_m is a c.s. covering of degree m which corresponds via Proposition 2.2 to the following c.s. covering of the special fibers (see Fig. 2):

3. *The Construction of the Map* τ: $V(X) \to (T)_G$

First, following Bloch [3], we introduce the following

DEFINITION 3.1. $V(X)$ is the kernel of the map

$$\text{Norm}: SK_1(X) \to k^\times,$$

which is induced by the norms: $\kappa(\mathfrak{p})^\times \to k^\times$ for all $\mathfrak{p} \in P$.

Let J be the Jacobian of X over k, and $T = \varprojlim_n J_n^{et}(k^{\text{sep}})$, where J_n^{et} is the etale part of the finite flat k-group scheme $J_n = \text{Ker}(J \to^n J)$. The Galois group $\text{Gal}(k^{\text{sep}}/k)$ acts on T, and we denote by $(T)_G$ its coinvariant. In this section we define the homomorphism

$$\tau: V(X) \to (T)_G,$$

which was defined first by Bloch in case $X(k) \neq \emptyset$ and $\text{ch}(k) = 0$. Our definition will not need those assumptions.

LEMMA 3.2. *There exists an exact sequence*

$$0 \to (T)_G \to \pi_1^{\text{ab}}(X) \to \text{Gal}(k^{\text{ab}}/k) \to 0.$$

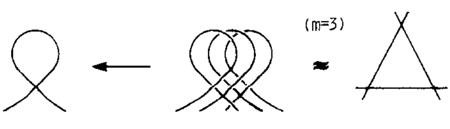


FIGURE 2

Proof. Considering the dual groups, it suffices to show the exactness of the following sequence

$$0 \to H^1(k, \mathbb{Q}/\mathbf{Z}) \to H^1(X, \mathbb{Q}/\mathbf{Z}) \to H^0(k, H^1(\bar{X}, \mathbb{Q}/\mathbf{Z})) \to 0, \tag{3.3}$$

where $\bar{X} = X \times_k k^{\mathrm{sep}}$. From the spectral sequence

$$H^q(k, H^r(\bar{X}, \mathbb{Q}/\mathbf{Z})) \Rightarrow H^{q+r}(X, \mathbb{Q}/\mathbf{Z}),$$

we have the exactness of (3.3) replacing the last 0 by $H^2(k, \mathbb{Q}/\mathbf{Z})$. Therefore it suffices to show that $H^2(k, \mathbb{Q}/\mathbf{Z}) = 0$, which follows from the Tate duality for local fields (cf. Serre [23, II, Sect. 5] and Shatz [24]).

LEMMA 3.4. *The following diagram is commutative*

$$\begin{tikzcd} SK_1(X) \arrow[r, "\mathrm{Norm}"] \arrow[d, "\sigma"'] & k^\times \arrow[d, "\rho_k"] \\ \pi_1^{\mathrm{ab}}(X) \arrow[r] & \mathrm{Gal}(k^{\mathrm{ab}}/k) \end{tikzcd}$$

where the right vertical arrow is the homomorphism coming from the local class field theory.

Proof. This follows from the definition of the map σ and the following commutative diagram in the local class field theory:

$$\begin{tikzcd} F^\times \arrow[r, "\mathrm{norm}"] \arrow[d, "\rho_F"'] & k^\times \arrow[d, "\rho_k"] \\ \mathrm{Gal}(F^{\mathrm{ab}}/F) \arrow[r] & \mathrm{Gal}(k^{\mathrm{ab}}/k) \end{tikzcd}$$

where F/k is a finite extension of local fields.

Now, by Definition 3.1 and Lemmas 3.2 and 3.4, we obtain the desired homomorphism $\tau\colon V(X) \to (T)_G$.

PROPOSITION 3.5. *The quotient group of $(T)_G$ by the closure of the image of τ is isomorphic to $\hat{\mathbf{Z}}^r$, where r is the rank of X over k (cf. Definition 2.5).*

Proof. In view of Theorem 2.6, Proposition 3.5 follows from the following commutative diagram, together with the fact that the image of Norm: $SK_1(X) \to k^\times$ has finite index in $k^\times$.

$$\begin{tikzcd} 0 \arrow[r] & V(X) \arrow[r] \arrow[d, "\tau"] & SK_1(X) \arrow[r, "\mathrm{Norm}"] \arrow[d, "\sigma"] & k^\times \arrow[d, "\rho_k"] & \\ 0 \arrow[r] & (T)_G \arrow[r] & \pi_1^{\mathrm{ab}}(X) \arrow[r] & \mathrm{Gal}(k^{\mathrm{ab}}/k) \arrow[r] & 0. \end{tikzcd}$$

4. *The Finiteness Theorem*

First we introduce some notations. For a prime number l, put $T_l = \varprojlim_n J^{et}_{l^n}(k^{\mathrm{sep}})$. By definition, T_l is the l-part of T (cf. Notations). Let $(T)'_G$ denote $(T)_G$ if $\mathrm{ch}(k)=0$, and $\prod_{l \neq \mathrm{ch}(k)} (T_l)_G$ if $\mathrm{ch}(k)>0$. Let τ' (resp. τ_l for a prime number l) denote the composite homomorphism

$$V(X) \xrightarrow{\tau} (T)_G \to (T)'_G \text{ (resp. } V(X) \xrightarrow{\tau} (T)_G \to (T_l)_G),$$

where the second homomorphism is the natural surjection to the prime-to-$\mathrm{ch}(k)$ part (resp. the l-part) of $(T)_G$. The purpose of this section is to prove the following

THEOREM 4.1. *The image of the map*

$$\tau' \colon V(X) \to (T)'_G$$

is finite.

Remark 4.2. Even if $p = \mathrm{ch}(k)$ is positive, the image of $\tau\colon V(X) \to (T)_G$ seems to be finite. But the author cannot prove the finiteness of the image of $\tau_p\colon V(X) \to (T_p)_G$.

The following result is a generalization of Bloch [3], Theorem 2.4 and Example 2.7.

COROLLARY 4.3. *Suppose* $\mathrm{ch}(k)=0$. *Then the image of* $\tau\colon V(X) \to (T)_G$ *is finite and the cokernel is isomorphic to* $\hat{\mathbf{Z}}^r$, *where* r *is the rank of* X *over* k. *In particular,* $(T)_G$ *is finite if and only if the rank of* X *over* k *is* 0.

Proof. This follows at once from Theorem 4.1 and Proposition 3.5.

We also have the following as an immediate consequence of Theorem 4.1 and Proposition 3.5.

COROLLARY 4.4. *For any prime* $l \neq \mathrm{ch}(k)$, *the image of* $\tau_l\colon V(X) \to (T_l)_G$ *is equal to the torsion part of* $(T_l)_G$, *and its cokernel is isomorphic to* $\mathbf{Z}_l^r$ *with* $r =$ *the rank of* X *over* k. *In particular, the rank of* $(T_l)_G$ *is independent of* $l \neq \mathrm{ch}(k)$.

Now we start the proof of Theorem 4.1. First, by the usual sort of norm argument, we may replace k by a finite extension.

LEMMA 4.5. *Replacing* k *by a finite extension, we may suppose the following*

(1) $X(k) \neq \varnothing$.

(2) *There exists a regular model of X over* $\operatorname{Spec} \mathcal{O}_k$, *such that the special fiber* $\mathcal{X}_s$ *satisfies the following conditions*; (a) $\mathcal{X}_s$ *is reduced*; (b) $\mathcal{X}_s$ *has only F-rational ordinary double points as singularities. Here F is the residue field of k*; (c) *any irreducible component of* $\mathcal{X}_s$ *is regular and geometrically irreducible over F.*

Proof. The first assertion is trivial. We prove the second assertion. Let g be the genus of X. If $g=0$, the assertion is trivial. If $g=1$, the assertion follows from the theory of Neron's minimal models of elliptic curves. Finally, if $g \geqslant 2$, the assertion follows from the theory of stable curves (cf. [6]).

From now on, we fix a regular model $\mathcal{X}$ of X such that the special fiber $\mathcal{X}_s$ satisfies the conditions of Lemma 4.5(2). Let C_i $(1 \leqslant i \leqslant n)$ be all the irreducible components of $\mathcal{X}_s$, and let K_i denote the completion of K along C_i, which is a complete discrete valuation field with residue field F_i, the function field of C_i. We can consider k as a subfield of K_i, and then k is algebraically closed in K_i. Furthermore, by Lemma 4.5(2)(a), a prime element π of k is also a prime element of K_i, and by (c), the residue field F of k is algebraically closed in F_i.

LEMMA 4.6. *Let $f\colon Y \to X$ be a finite etale covering, L the function field of Y. If f is trivial over any K_i, that is, $L \otimes_K K_i$ is isomorphic to the direct product of finite copies of K_i, then f is a* c.s. *covering.*

Proof. Let $\mathcal{Y}$ be the integral closure of $\mathcal{X}$ in L, and $\bar{f}\colon \mathcal{Y} \to \mathcal{X}$ the extension of f. First, by the assumption and S.G.A.X. Theorem 1.8, $\bar{f}$ is etale. Then the assumption implies that the restriction of $\bar{f}$ to the special fiber, $f_s\colon \mathcal{Y}_s \to \mathcal{X}_s$ is a c.s. covering. This implies our assertion.

Now, for the proof of Theorem 4.1, it suffices to show the following claims.

Claim 1. For any prime $l \neq \operatorname{ch}(k)$, the image of τ_l is finite.

Claim 2. For almost all primes l, the image of τ_l is trivial.

Let I_l be the closure of the image of τ_l. Since the quotient $(T_l)_G/I_l$ is a free $\mathbf{Z}_l$-module of finite rank (cf. Proposition 3.5), for the proof of the claims, it suffices to show the following assertions.

Claim 1′. For any prime $l \neq \operatorname{ch}(k)$, any $\mathbf{Z}_l$-homomorphism $(T_l)_G \to \mathbf{Z}_l$ is trivial on I_l.

Claim 2′. For almost all primes l, any homomorphism $(T_l)_G \to \mathbf{Z}/l\mathbf{Z}$ is trivial on I_l.

Now we fix a k-rational point $z \in X(k)$ (cf. Lemma 4.5(1)). Then z defines a section of the natural surjection

$$\pi_1^{\mathrm{ab}}(X) \to \operatorname{Gal}(k^{\mathrm{ab}}/k),$$

and we obtain a decomposition

$$\pi_1^{\mathrm{ab}}(X) \simeq (T)_G \times \mathrm{Gal}(k^{\mathrm{ab}}/k) \qquad \text{(cf. Lemma 3.2)}, \tag{4.7}$$

where $(T)_G$ is identified with the Galois group which classifies abelian etale coverings of X in which z splits completely. Through the decomposition (4.7), a $\mathbf{Z}_l$-homomorphism

$$\chi\colon (T_l)_G \to \mathbf{Z}_l$$
$$(\text{resp. } \chi\colon (T_l)_G \to \mathbf{Z}/l\mathbf{Z})$$

is identified with an element of $H^1(X, \mathbf{Z}_l)$ (resp. $H^1(X, \mathbf{Z}/l\mathbf{Z})$), and χ is trivial on I_l if and only if the corresponding covering of X is a c.s. covering. Hence, considering Lemma 4.6, Claim 1′, and Claim 2′ are equivalent to the following assertions.

Claim 1″. For any prime $l \neq \mathrm{ch}(k)$, the subgroup $H^1(k, \mathbf{Z}_l)$ of $H^1(X, \mathbf{Z}_l)$ maps *onto* the image of the natural homomorphism

$$H^1(X, \mathbf{Z}_l) \to \prod_i H^1(K_i, \mathbf{Z}_l).$$

Claim 2″. For almost all primes l, the subgroup $H^1(k, \mathbf{Z}/l\mathbf{Z})$ of $H^1(X, \mathbf{Z}/l\mathbf{Z})$ maps *onto* the image of the natural homomorphism

$$H^1(X, \mathbf{Z}/l\mathbf{Z}) \to \prod_i H^1(K_i, \mathbf{Z}/l\mathbf{Z}).$$

The following lemma is a key to the proof of these claims.

LEMMA 4.8. *Let M be a complete discrete valuation field with residue field M_0, and N be a subfield.*

(1) *Suppose that the following conditions are satisfied:*

(i) *N is complete under the restriction of the discrete valuation of M.*

(ii) *A prime element of N is a prime element of M.*

(iii) *N is algebraically closed in M.*

Let L be a finite abelian extension of M of degree prime to $\mathrm{ch}(M_0)$. *Then there exists an unramified abelian extension L_u of M and an abelian extension L_c of N such that*

$$L \subset L_u L_c.$$

(2) *Suppose that* $\mathrm{ch}(M)=0$ *and* $p=\mathrm{ch}(M_0)$ *is positive, and that the conditions* (i), (ii) *and the following condition* (iii′) *are satisfied*:

(iii′) *The residue field of* N *is the maximum perfect subfield of* M_0.

Then, for any $\mathbf{Z}_p$*-extension* L *of* M, *there exists an unramified abelian extension* L_u *of* M *and an abelian extension* L_c *of* N *such that* $L \subset L_u L_c$.

Proof. Lemma 4.8(1) is due to Ihara [10] (cf. also Miki [17, Proposition 7]), and Lemma 4.8(2) is a theorem of Miki [17].

Now we prove Claim 1″ at first. Fix a prime $l \neq \mathrm{ch}(k)$. Let $\chi \in H^1(X, \mathbf{Z}_l)$ and χ_i be its image in $H^1(K_i, \mathbf{Z}_l) \pm$ or $1 \leqslant i \leqslant n$. By Lemma 4.8 we can write

$$\chi_i = \varphi_i + \psi_i,$$

with $\varphi_i \in H^1(k, \mathbf{Z}_l)$ and $\psi_i \in H^1(F_i, \mathbf{Z}_l)$, where we identify $H^1(k, \mathbf{Z}_l)$ and $H^1(F_i, \mathbf{Z}_l)$ with their images in $H^1(K_i, \mathbf{Z}_l)$.

Step 1. We can write

$$\varphi_i = \varphi + \eta_i,$$

where $\varphi \in H^1(k, \mathbf{Z}_l)$ does not depend on i, and $\eta_i \in H^1(F, \mathbf{Z}_l)$.

In fact, let C_1 and C_2 be two irreducible components of $\mathscr{X}_s$ which intersect on a closed point x of $\mathscr{X}_s$. Let R_x be the completion of the local ring of $\mathscr{X}$ at x, and put $A_x = R_x[1/\pi]$, where π is a prime element of k, and K_x the quotient field of R_x. We have isomorphisms

$$R_x \simeq \mathcal{O}_k[[\mathrm{X}, \mathrm{Y}]]/(\mathrm{XY}-\pi),$$

$$A_x \simeq \mathcal{O}_k[[\mathrm{X}, \mathrm{Y}]]/(\mathrm{XY}-\pi)\left[\frac{1}{\pi}\right],$$

where the prime ideal (X) (resp. (Y)) of R_x corresponds to the component C_1 (resp. C_2). Let $K_{x.i}$ be the completion of K_x along C_i for $i=1, 2$. We can consider K_i and k as subfields of $K_{x.i}$, and π is a prime element of both K_i and $K_{x.i}$. The residue field $F_{x.i}$ of $K_{x.i}$ is the completion of F_i at x.

Let $\varphi_{x.i}$ (resp. $\psi_{x.i}$) be the image of φ_i (resp. ψ_i) under the restriction map

$$H^1(K_i, \mathbf{Z}_l) \rightarrow H^1(K_{x.i}, \mathbf{Z}_l),$$

and let $\bar{\psi}_{x.i}$ be the image of ψ_i under the restriction map

$$H^1(F_i, \mathbf{Z}_l) \rightarrow H^1(F_{x.i}, \mathbf{Z}_l).$$

Then $\psi_{x.i}$ is the image of $\bar{\psi}_{x.i}$ under the inclusion

$$H^1(F_{x.i}, \mathbf{Z}_l) \rightarrow H^1(K_{x.i}, \mathbf{Z}_l).$$

By the higher local class field theory (cf. Chap. I, Theorem 3.1), we identify $\varphi_{x.i}$ and $\psi_{x.i}$ with the corresponding elements of $\mathrm{Hom}_c(K_2(K_{x.i}), \mathbf{Z}_l)$, where Hom_c means continuous homomorphisms. On the other hand, by the local class field theory, we identify $\bar{\psi}_{x.i}$ with an element of $\mathrm{Hom}_c(F_{x.i}^{\times}, \mathbf{Z}_l)$, and φ_i with an element of $\mathrm{Hom}_c(k^{\times}, \mathbf{Z}_l)$. By the class field theory of $K_{x.i}$, we have a commutative diagram (cf. Chap. I, Theorem 3.1(2)),

$$\begin{tikzcd} K_2(K_{x.i}) \arrow[rr, "\partial^i"] \arrow[dr, "\psi_{x.i}"'] & & F_{x.i}^{\times} \arrow[dl, "\bar{\psi}_{x.i}"] \\ & \mathbf{Z}_l & \end{tikzcd}, \tag{4.9}$$

where ∂^i is the boundary map in K-theory. On the other hand, Kato [3] defined the homomorphism

$$\mathrm{Res}^i\colon K_2(K_{x.i}) \to k^{\times},$$

such that we have a commutative diagram

$$\begin{tikzcd} K_2(K_{x.i}) \arrow[r, "-\mathrm{Res}^i"] \arrow[d] & k^{\times} \arrow[d] \\ \mathrm{Gal}(K_{x.i}^{\mathrm{ab}}/K_{x.i}) \arrow[r] & \mathrm{Gal}(k^{\mathrm{ab}}/k), \end{tikzcd}$$

where the vertical arrows come from the class field theory of $K_{x.i}$ and k. Hence we have a commutative diagram

$$\begin{tikzcd} K_2(K_{x.i}) \arrow[rr, "-\mathrm{Res}^i"] \arrow[dr, "\varphi_{x.i}"'] & & k^{\times} \arrow[dl, "\varphi_i"] \\ & \mathbf{Z}_l & \end{tikzcd} \tag{4.10}$$

Now, let χ_x be the restriction of χ to $H^1(K_x, \mathbf{Z}_l)$. By definition it is contained in the subgroup $H^1(\mathrm{Spec}\, A_x, \mathbb{Z}_l)$. We apply Chap. I, Proposition 3.4 to χ_x, and obtain an equation, for any $a \in K_2(A_x)$,

$$\varphi_{x.1}(j_1(a)) + \psi_{x.1}(j_1(a)) + \varphi_{x.2}(j_2(a)) + \psi_{x.2}(j_2(a)) = 0, \tag{4.11}$$

where $j_i\colon K_2(A_x) \to K_2(K_{x.i})$ $(i = 1, 2)$ is the natural homomorphism. Take the element $a = \{u, X\} \in K_2(A_x)$, where $u \in U_k^1$. Then, by the calculation of Kato [14], we have

$$\mathrm{Res}^1(j_1(a)) = u \in k^{\times}, \qquad \partial^1(j_1(a)) = \bar{u} = 1 \in F^{\times},$$

$$\mathrm{Res}^2(j_2(a)) = u^{-1} \in k^{\times}, \qquad \partial^2(j_2(a)) = 1 \in F^{\times}.$$

Hence, from (4.9), (4.10) and (4.11), we have

$$\varphi_1(u)=\varphi_2(u) \qquad \text{for} \quad u\in U_k^1.$$

Since $\mathbf{Z}_l$ is torsion free, this implies that φ_1 and φ_2 coincide on the subgroup $\mathcal{O}_k^\times$. By the connectedness of $\mathscr{X}_s$, this holds for any pair φ_i and φ_j. This completes the proof of Step 1.

By Step 1 we can write $\chi_i=\varphi+\psi_i$, where $\varphi\in H^1(k,\mathbf{Z}_l)$ does not depend on i, and $\psi_i\in H^1(F_i,\mathbf{Z}_l)$. For the proof of Claim 1″, replacing χ by $\chi-\varphi$, we may suppose $\varphi=0$. Then χ_i $(=\psi_i)$ is contained in the subgroup $H^1(F_i,\mathbf{Z}_l)$ of $H^1(K_i,\mathbf{Z}_l)$. By S.G.A.X. Theorem 3.1, this implies that the $\mathbf{Z}_l$-etale covering of X (cf. Notations) corresponding to χ extends to a *$\mathbf{Z}_l$-etale covering $\mathscr{Y}$ of $\mathscr{X}$*, and then χ_i corresponds to the covering of C_i induced by $\mathscr{Y}$. Hence χ_i is contained in the subgroup $H^1(C_i,\mathbf{Z}_l)$ of $H^1(F_i,\mathbf{Z}_l)$. On the other hand, by the classical class field theory, we know that the kernel of the natural homomorphism $\pi_1^{\mathrm{ab}}(C_i)\to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is finite (By Lemma 4.5(2) F is the constant field of C_i). Hence χ_i is in the image of $H^1(F,\mathbf{Z}_l)$ under the inclusions

$$\begin{tikzcd} & H^1(F_i,\mathbf{Z}_l) \arrow[dr, hook] & \\ H^1(F,\mathbf{Z}_l) \arrow[ur, hook] \arrow[dr, hook] & & H^1(K_i,\mathbf{Z}_l). \\ & H^1(k,\mathbf{Z}_l) \arrow[ur, hook] & \end{tikzcd} \tag{4.12}$$

Now the following suffices to complete the proof of Claim 1″.

Step 2. All χ_i come from the same element of $H^1(F,\mathbf{Z}_l)$.

In fact, let x, C_1, C_2, etc. be as in the proof of Step 1, and let $\chi_{x.i}$ $(i=1,2)$ be the restriction of χ_i to $H^1(K_{x.i},\mathbf{Z}_l)$. We identify $\chi_{x.i}$ with an element of $\mathrm{Hom}_c(K_2(K_{x.i}),\mathbf{Z}_l)$ by Chap. I, Theorem 3.1. On the other hand, when we view χ_i as an element of $H^1(F,\mathbf{Z}_l)$ via (4.12), χ_i is identified with an element of $\mathrm{Hom}_c(k^\times/\mathcal{O}_k^\times,\mathbf{Z}_l)$ $(=\varprojlim_n \mathrm{Hom}(k^\times/\mathcal{O}_k^\times,\mathbf{Z}/l^n\mathbf{Z}))$. Then, as the proof of Step 1, we have a commutative diagram

$$\begin{tikzcd} K_2(K_{x.i}) \arrow[r, "-\mathrm{Res}^i"] \arrow[dr, "\chi_{x.i}"'] & k^\times \arrow[r] & k^\times/\mathcal{O}_k^\times \arrow[dl, "\chi_i"] \\ & \mathbf{Z}_l & \end{tikzcd} \tag{4.13}$$

and an equation, for any $a \in K_2(A_x)$,

$$\chi_{x.1}(j_1(a)) + \chi_{x.2}(j_2(a)) = 0. \tag{4.14}$$

Take the element $a = \{\pi, X\} \in K_2(A_x)$. By the calculation of Kato [14], we have $\mathrm{Res}^1(j_1(a)) = \pi$, and $\mathrm{Res}^2(j_2(a)) = \pi^{-1}$. Hence, by (4.13) and (4.14), we have

$$\chi_1(\bar{\pi}) + \chi_2(\bar{\pi}^{-1}) = 0,$$

where $\bar{\pi}$ and $\bar{\pi}^{-1}$ are the classes of π and π^{-1} in $k^\times/\mathcal{O}_k^\times$. This implies that $\chi_1 = \chi_2$ as an element of $H^1(F, \mathbf{Z}_l)$, and by the connectedness of $\mathcal{X}_s$, this suffices to complete the proof.

Now, to complete the proof of Theorem 4.1, there remains the proof of Claim 2″. So let $\chi \in H^1(X, \mathbf{Z}/l\mathbf{Z})$ and χ_i the restriction of χ to $H^1(K_i, \mathbf{Z}/l\mathbf{Z})$ for each i. We prove that all χ_i come from the same element of $H^1(k, \mathbf{Z}/l\mathbf{Z})$ under the following assumptions.

(A) $l \neq \mathrm{ch}(F)$ and $l \nmid q-1$, where $q = \# F$.

(B) Let J_i be the Jacobian of C_i over F for each i. Then $l \nmid \# J_i(F)$ for any i.

First, by Lemma 4.8, we can write $\chi_i = \varphi_i + \psi_i$, with $\varphi_i \in H^1(k, \mathbf{Z}/l\mathbf{Z})$ and $\psi_i \in H^1(F_i, \mathbf{Z}/l\mathbf{Z})$ for each i. Then, by the same argument as Step 1 of the proof of Claim 1″, we can write $\chi_i = \varphi + \psi_i$, where $\varphi \in H^1(k, \mathbf{Z}/l\mathbf{Z})$ does not depend on i, and $\psi_i \in H^1(F_i, \mathbf{Z}/l\mathbf{Z})$ for each i. The only difference from Step 1 is the following: We can prove at first the coincidence of φ_i and φ_j only on the subgroup U_k^1 of $k^\times$. But by the assumption (A), we can conclude that φ_i and φ_j coincide on $\mathcal{O}_k^\times$, for $\mathcal{O}_k^\times/U_k^1 \simeq \mathbf{Z}/(q-1)\mathbf{Z}$.

Now, replacing χ with $\chi - \varphi$, we can suppose $\varphi = 0$, and using S.G.A.X. Theorem 3.1, we can conclude that χ_i comes from $H^1(C_i, \mathbf{Z}/l\mathbf{Z})$. On the other hand, by the class field theory of curves over finite fields, we have the exact sequence

$$0 \to H^1(F, \mathbf{Z}/l\mathbf{Z}) \to H^1(C_i, \mathbf{Z}/l\mathbf{Z}) \to \mathrm{Hom}(J_i(F), \mathbf{Z}/l\mathbf{Z}) \to 0.$$

Hence, by the assumption (B), we have

$$H^1(F, \mathbf{Z}/l\mathbf{Z}) \xrightarrow{\sim} H^1(C_i, \mathbf{Z}/l\mathbf{Z}) \qquad \text{for any } i.$$

Consequently, χ_i comes from $H^1(F, \mathbf{Z}/l\mathbf{Z})$ for any i. Finally, by the same argument as Step 2 of the proof of Claim 1″, we can conclude that all χ_i come from the same element of $H^1(F, \mathbf{Z}/l\mathbf{Z})$, and this suffices to complete the proof.

5. *The Kernel of σ and τ*

The purpose of this section is to determine the kernel of the prime-to-ch(k) part of σ and τ. First we introduce some notations: Let Q be the set of all prime numbers other than ch(k), and fix a subset (not necessarily finite) S of Q. Let D^S (resp. E^S) be the maximal S-divisible subgroup (that is, the maximal subgroup which is divisible by any prime $l \in S$) of $V(X)$ (resp. $SK_1(X)$). Let $\pi_1^{ab}(X)(S)$ (resp. $(T_S)_G$) denote the S-part of $\pi_1^{ab}(X)$ (resp. $(T)_G$) (cf. Notations), and we denote by

$$\sigma^S\colon SK_1(X) \to \pi_1^{ab}(X)(S)$$
$$(\text{resp. } \tau^S\colon V(X) \to (T_S)_G)$$

the composite homomorphism of σ (resp. τ) and the natural surjection $\pi_1^{ab}(X) \to \pi_1^{ab}(X)(S)$ (resp. $(T)_G \to (T_S)_G$).

The main result of this section is the following

THEOREM 5.1. *The kernel of the map σ^S (resp. τ^S) is equal to E^S (resp. D^S).*

COROLLARY 5.2. (1) *For any prime $l \neq$ ch(k), the quotient of $V(X)$ by its maximal l-divisible subgroup is finite, and isomorphic to the torsion part of $(T_l)_G$.*

(2) *Suppose* ch$(k) = 0$. *Then the quotient of $V(X)$ by its maximal divisible subgroup is finite, and isomorphic to the torsion part of $(T)_G$.*

Corollary 5.2 follows at once from Theorem 5.1 and Corollaries 4.3 and 4.4.

For the proof of Theorem 5.1, we need the following

LEMMA 5.3. *Let n be an integer >0 prime to* ch(k). *Then the map $\sigma\colon SK_1(X) \to \pi_1^{ab}(X)$ induces the injection*

$$SK_1(X)/nSK_1(X) \hookrightarrow \pi_1^{ab}(X)/n\pi_1^{ab}(X).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
K_2(K) & \to & \bigoplus_{\mathfrak{p}\in P} \kappa(\mathfrak{p})^\times/(\kappa(\mathfrak{p})^\times)^n & \to & SK_1(X)/nSK_1(X) & \to & 0 \\
\downarrow{\scriptstyle h_K^2} & & \wr\downarrow{\scriptstyle \oplus h^1_{\kappa(\mathfrak{p})}} & & & & \\
H^2(K, \mu_n^{\otimes 2}) & \to & \bigoplus_{\mathfrak{p}\in P} H^1(\kappa(\mathfrak{p}), \mu_n) & \to & H^3(X, \mu_n^{\otimes 2}) & &
\end{array}$$

Here the upper sequence is the sequence defining $SK_1(X)$ modulo n, and the lower sequence is the localization sequence in etale cohomology theory of X. The map $h^1_{\kappa(\mathfrak{p})}$ is the well-known isomorphism coming from Kummer theory of $\kappa(\mathfrak{p})$, and the map h^2_K is the Galois symbol on K (cf. Tate [26]). By the result of Mercuriev and Suslin (cf. [18]), we know the surjectivity of h^2_K. The commutativity of the left square is well known. Hence, from the diagram, we obtain an injective homomorphism

$$SK_1(X)/nSK_1(X) \hookrightarrow H^3(X, \mu_n^{\otimes 2}). \tag{5.4}$$

On the other hand, by the combination of Poincaré duality for $\bar{X} = X \times_k k^{\mathrm{sep}}$ (cf. S.G.A.4, XVIII) and the Tate duality for Galois cohomology of k (cf. Serre [23, II, Sect. 5]), we have an isomorphism

$$H^3(X, \mu_n^{\otimes 2}) \simeq H^1(X, \mathbf{Z}/n\mathbf{Z})^* \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(X)/n\pi_1^{\mathrm{ab}}(X), \tag{5.5}$$

where * means the dual of finite abelian groups. By the definition of the map σ and Chap. I, Theorem 3.1(2), we conclude that the composite of (5.5) and (5.4) is nothing other than the map σ modulo n. Hence the proof is completed.

Now we prove Theorem 5.1. Let H (resp. J) be the kernel of the map σ^S (resp. τ^S). By definition, we have $J = H \cap V(X)$. Since $\pi_1^{\mathrm{ab}}(X)(S)$ and $(T_S)_G$ contain no nonzero S-divisible element, we have the inclusions $E^S \hookrightarrow H$ and $D^S \hookrightarrow J$. Hence it suffices to show that H and J are S-divisible. First we prove it for J. By Lemma 5.3 we have $H = \bigcap_{n \mid S} nSK_1(X)$, where $n \mid S$ ("n divides S") means that all prime divisors of n are in S, and n runs over all such integers. Therefore we have $J = \bigcap_{n \mid S} J_n$ with $J_n = nSK_1(X) \cap V(X)$. Consequently the following claim suffices to prove our assertion:

Claim. There exists an integer $N \mid S$ such that we have

$$J_N = J = \bigcap_{n \mid S} J_n.$$

Proof. First we note the following fact: Let M be the order of the group of the roots of unit in k whose orders divide S. Then, for any integer $n \mid S$, we have $J_{nM} \hookrightarrow nV(X)$. Let M' be the order of the image of the map τ^S, which is finite by Theorem 4.1, and put $N = MM'$. By definition, we have the inclusions $J_N \hookrightarrow M'V(X)$ and $M'V(X) \hookrightarrow J$. Hence N satisfies the condition of the Claim.

Now we prove that H is S-divisible. Since we proved $J = H \cap V(X)$ is S-divisible, it suffices to show that the image of H under Norm: $SK_1(X) \to k^\times$ is S-divisible. We consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & J \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & U \arrow[d] \\
0 \arrow[r] & V(X) \arrow[r] \arrow[d] & SK_1(X) \arrow[r, "\mathrm{Norm}"] \arrow[d] & k^\times \arrow[d] \\
0 \arrow[r] & (T_S)_G \arrow[r] & \pi_1^{ab}(X)(S) \arrow[r] & \mathrm{Gal}(k^{ab}/k)(S),
\end{tikzcd}$$

where $\mathrm{Gal}(k^{ab}/k)(S)$ is the S-part of $\mathrm{Gal}(k^{ab}/k)$ (cf. Notations) and U is the kernel of $\rho_k^S: k^\times \to \mathrm{Gal}(k^{ab}/k)(S)$ which comes from the local class field theory. Then our assertion follows from the local class field theory of k, together with the fact the image of Norm is open and has finite index in $k^\times$.

6. *The Rank of X*

In this section, we give some properties concerning the rank of X over k (cf. Definition 2.5). For a finite extension k'/k, we denote by $r_{k'}$ the rank of $X \times_k k'$ over k'. From Corollary 4.3 (or Corollary 4.4), we can see

$$r \leqslant r_{k'} \leqslant 2g,$$

where g denotes the genus of X. Hence, when k' runs over all finite extensions of k, there exists the maximum value of $r_{k'}$.

DEFINITION 6.1. We call the maximum value $r_{k'}$ *the geometrical rank of* X and denote it by $\bar{r}$.

Now we introduce some notations. Let F be the residue field of k, J the Jacobian of X over k, $\tilde{J}$ the Neron's minimal model of J over $\mathrm{Spec}\,\mathcal{O}_k$, J_s the special fiber of $\tilde{J}$, and J_s^0 the connected component of J_s of identity. The following fact is well known from the general theory of commutative algebraic groups: J_s^0 is an extension of an abelian variety A over F by a linear algebraic group H over F, and there exists an isomorphism $H \simeq S \times U$ over F, where S is a torus and U is unipotent.

Now the main result of this section is the following

THEOREM 6.2. (1) *The rank r of X over k is equal to the rank of the group of F-rational characters of S.*

(2) *Let $\bar{r}$ be the geometrical rank of X. We have an inequality*

$$\bar{r} \leqslant g,$$

where g is the genus of X. The equality $\bar{r}=g$ holds if and only if there exists a finite extension k'/k and a regular model $\mathscr{X}$ of X over Spec $\mathcal{O}_{k'}$, *such that the special fiber $\mathscr{X}_s$ is reduced and has as singularities only ordinary double points and all irreducible components of $\mathscr{X}_s$ are smooth rational curves. The equality $\bar{r}=0$ holds if and only if there exists a finite extension k'/k over which J has good reduction.*

Proof of (1). In the following proof, we use the following notations: For a commutative algebraic group Z over a field L, and for a prime number l, we denote by $Z(l)$ the l-torsion part of the abelian group $Z(L^{\mathrm{sep}})$.

Now, let $G=\mathrm{Gal}(k^{\mathrm{sep}}/k)$ and $H=\mathrm{Gal}(\bar{F}/F)$. Fix a prime $l\neq \mathrm{ch}(F)$, and let T_l be the Tate module of J. Put $M=\mathrm{Hom}(S(l), \mathbf{G}_m(l))$. Then, M is a free $\mathbf{Z}_l$-module of finite rank, and H acts on M as follows: For $\chi\in M$, $x\in S(l)$, and $g\in H$,

$$\chi^g(x)=g\cdot\chi(g^{-1}\cdot x).$$

Moreover, if we denote by N the character group of $\bar{S}=S\times_F\bar{F}$, we have an isomorphism $M\simeq N\otimes_{\mathbf{Z}}\mathbf{Z}_l$. Hence, by Corollary 4.4, for the proof it suffies to show that

$$\text{the rank of } (T_l)_G=\text{the rank of } M^H.$$

Recall that there exists a perfect pairing called "e_l-pairing";

$$J(l)\otimes T_l\to \mathbf{G}_m(l), \tag{6.3}$$

where $J(l)$ (resp. T_l) is endowed with discrete topology (resp. compact topology as a pro-l-group). Let I be the inertia subgroup of G. We have $G/I \xrightarrow{\sim} H$. Since I acts trivially on $\mathbf{G}_m(l)$, (6.3) induces a perfect pairing

$$J(l)^I\otimes (T_l)_I\to \mathbf{G}_m(l). \tag{6.4}$$

On the other hand, by [25], we have an isomorphism

$$J(l)^I\simeq J_s(l). \tag{6.5}$$

Let f be the Frobenius map over F, and $q=\#F$. By definition, there is an isomorphism $H\xrightarrow{\sim}\hat{\mathbf{Z}}$ which sends f to $1\in\hat{\mathbf{Z}}$. Since f acts on $\mathbf{G}_m(l)$ as the qth power homomorphism, from (6.4) and (6.5), we have a perfect pairing

$$J_s(l)^{f-q}\otimes (T_l)_G\to \mathbf{G}_m(l), \tag{6.6}$$

where $J_s(l)^{f-q}$ denotes the kernel of $f-q$: $J_s(l)\to J_s(l)$. On the other hand, by definition, we have a perfect pairing

$$S(l)\otimes M\to \mathbf{G}_m(l),$$

which induces a perfect pairing

$$S(l)^{f-q} \otimes M_H \to \mathbf{G}_m(l). \tag{6.7}$$

Consequently, combining (6.6) and (6.7), the following claims are sufficient to complete the proof.

Claim 1. The rank of M^H = the rank of M_H.

Claim 2. $S(l)^{f-q}$ has finite index in $J_s(l)^{f-q}$.

Proof of Claim 1. Since H is topologically generated by f, we have the following exact sequence

$$0 \to M^H \to M \xrightarrow{f-1} M \to M_H \to 0.$$

Hence we have an exact sequence

$$0 \to M^H \otimes_{\mathbf{Z}_l} \mathbb{Q}_l \to M \otimes_{\mathbf{Z}_l} \mathbb{Q}_l \xrightarrow{f-1} M \otimes_{\mathbf{Z}_l} \mathbb{Q}_l \to M_H \otimes_{\mathbf{Z}_l} \mathbb{Q}_l \to 0.$$

Hence we can see $\dim_{\mathbb{Q}_l} M^H \otimes_{\mathbf{Z}_l} \mathbb{Q}_l = \dim_{\mathbb{Q}_l} M_H \otimes_{\mathbf{Z}_l} \mathbb{Q}_l$.

Proof of Claim 2. First, by definition, we have an exact sequence

$$0 \to S(l) \to J_s^0(l) \to A(l) \to 0.$$

Taking the kernels of the map $f - q$, we obtain the exact sequence

$$0 \to S(l)^{f-q} \to J_s^0(l)^{f-q} \to A(l)^{f-q}.$$

Hence it suffices to show that $A(l)^{f-q}$ is finite. Let $T_l B$ be the Tate module of the dual abelian variety B of A. By Deligne [5], the cokernel of

$$f - q \colon T_l B \to T_l B$$

is finite. This cokernel is isomorphic to the dual of $A(l)^{f-q}$ by the e_l-pairing $A(l) \otimes T_l B \to \mathbf{G}_m(l)$. Hence the proof is completed.

Proof of Theorem 6.2(2). The inequality $r \leqslant g$ follows at once from (1). We prove the condition for the equalities. First we prove the "only if" part. Put $S = \operatorname{Spec} \mathcal{O}_k$ and $s = \operatorname{Spec} F$. For the proof, we may replace k by a finite extension. Then, by Lemma 4.5, we may suppose that there exists a regular model $\mathcal{X}$ of X over S such that the special fiber $\mathcal{X}_s$ satisfies the conditions (a), (b), and (c) of Lemma 4.5. Let $\tilde{J}^0$ be the open subgroup scheme of $\tilde{J}$ with the special fiber J_s^0. By Raynaud [1], $\tilde{J}^0/S$ represents the functor $\mathbf{Pic}^0\mathcal{X}/S$. Hence J_s^0 represents the functor $\mathbf{Pic}^0\mathcal{X}_s/s$. Let C_i $(1 \leqslant i \leqslant n)$ be all the irreducible components of $\mathcal{X}_s$, and let J_i be the Jacobian of C_i over F.

Let r denote the rank of X over k. By definition, it is the rank of $H_1(\Gamma, \mathbf{Z})$, where Γ is the graph of $\mathscr{X}_s$. We can see easily that $\mathbf{Pic}^0\mathscr{X}_s/s$ is represented by a scheme which is an extension of $J_1 \times \cdots \times J_n$ by $\mathbf{G}_m \times \cdots \times \mathbf{G}_m$ (r-times direct product of $\mathbf{G}_m$). Consequently, comparing the dimensions of these schemes J_s^0 and $\mathbf{Pic}^0\mathscr{X}_s/s$, we obtain

$$\sum_{i=1}^{n} g(C_i) + r = g,$$

where $g(C_i)$ is the genus of C_i. This completes the proof.

Now, the proof of the "if" part is similar, and left to the readers.

7. *Nonabelian c.s. coverings*

In this section, we investigate nonabelian c.s. coverings. We fix a regular model $\mathscr{X}$ of X over Spec $\mathcal{O}_k$ such that the reduced part $(\mathscr{X}_s)_{\text{red}}$ of the special fiber $\mathscr{X}_s$ has as singularities only ordinary double points (cf. Sect. 2). Let Γ be the graph of $(\mathscr{X}_s)_{\text{red}}$ (cf. Definition 2.3), and we denote by $\pi_1(\Gamma)$ the topological fundamental group of Γ. It is a free group of rank r = the rank of X over k (cf. Definition 2.5).

The main result of this section is the following

THEOREM 7.1. (1) *Let L be a profinite separable extension of K in which almost all $\mathfrak{p} \in P$ split completely. Then all $\mathfrak{p} \in P$ split completely in the extension.*

(2) *Let $\tilde{K}$ be the maximum unramified Galois extension of K in which all $\mathfrak{p} \in P$ split completely. Then there exists an isomorphism* $\mathrm{Gal}(\tilde{K}/K) \simeq \widehat{\pi_1(\Gamma)}$, *where* $\hat{}$ *means the profinite completion with respect to normal subgroups of finite index.*

Proof of (1). We may assume that L/K is a finite Galois extension. Let Y be the projective smooth curve with the function field L and $f: Y \to X$ be the finite covering corresponding to the extension L/K. Let $\mathscr{Y}$ be the integral closure of $\mathscr{X}$ in L and $\bar{f}: \mathscr{Y} \to \mathscr{X}$ be the extension of f. By the assumption there exists a finite subset S of P such that if $\mathfrak{p} \in P$ is not in S, $\mathfrak{p}$ splits completely in L/K. We consider S as a closed subscheme of X with the reduced structure, and put $\mathscr{S}$ the closure of S in $\mathscr{X}$ and $\mathscr{R} = (\mathscr{X}_s)_{\text{red}} \cup \mathscr{S}$. The main point to prove is the following

Claim. $\bar{f}$ is an etale covering.

In fact, by definition, $\bar{f}$ is etale over the open subscheme $\mathscr{X} - \mathscr{R}$. Hence, by S.G.A.X. Theorem 3.1, it suffices to prove that $\bar{f}$ is etale at any generic point of $\bar{f}^{-1}(\mathscr{R})$. Thus, let η be an irreducible component of $\bar{f}^{-1}(\mathscr{R})$ and $\bar{\eta}$ be the irreducible component of $\mathscr{R}$ lying under η. Let $G = \mathrm{Gal}(L/K) =$

$\mathrm{Aut}(Y/X)=\mathrm{Aut}(\mathscr{Y}/\mathscr{X})$. We define D, the decomposition group of η, to be the subgroup of G consisting of all elements which fix the divisor η, and I, the inertia group of η, to be the subgroup of D consisting of all elements acting trivially on the function field $\kappa(\eta)$ of η. Let Y_I (resp. Y_D) be the finite covering of X corresponding to I (resp. D), and let $\mathscr{I}$ (resp. $\mathscr{D}$) be the integral closure of $\mathscr{X}$ in the function field of Y_I (resp. Y_D). Let η_i (resp. η_d) be the divisor of $\mathscr{I}$ (resp. $\mathscr{D}$) lying under η. Then we know the following facts:

(i) The finite covering $\mathscr{D}\to\mathscr{X}$ is etale at η_d and we have $\kappa(\bar{\eta})\simeq\kappa(\eta_d)$.

(ii) The finite covering $\mathscr{I}\to\mathscr{D}$ is etale at η_i, and $\kappa(\eta_i)$ is the separable closure of $\kappa(\bar{\eta})$ in $\kappa(\eta)$.

(iii) I is an extension of a cyclic group of order prime to p by a p-group, where p is the characteristic of the residue field of k.

By (i) and (ii), to prove $\tilde{f}$ is etale at η, it suffices to show that the covering $\mathscr{Y}\to\mathscr{I}$ is etale at η. On the other hand, by (iii), the covering $Y\to Y_I$ factors through a finite number of cyclic coverings. Hence, by the induction on the degree, we may suppose it is a cyclic covering. Let x be a closed point of $\mathscr{I}$ lying on η_i, and R_x be the completion of the local ring of $\mathscr{I}$ at x. By definition, R_x is a two-dimensional normal complete local ring. By the assumption, almost all prime ideals of height 1 in R_x split completely in the extension $\tilde{R}=R_x\otimes_{\mathscr{I}}\mathscr{Y}$ of R_x. Hence, by Chap. I, Proposition 3.3, we conclude that all prime ideals of height 1 in R_x split completely in the extension. This implies that the covering $\mathscr{Y}\to\mathscr{I}$ is etale at η, and the proof of the claim is completed.

Now that we have proved the claim, we can easily see that the restriction of $\tilde{f}$ to the special fibers is a c.s. covering, and Theorem 7.1(1) follows at once from this fact.

Proof of (2). It suffices to show the following:

Claim 1. There exists an isomorphism

$$\mathrm{Gal}(\tilde{K}/K)\simeq\tilde{\pi}_1(\mathscr{X}_s),$$

where $\tilde{\pi}_1(\mathscr{X}_s)$ is the quotient group of $\pi_1(\mathscr{X}_s)$ classifying finite c.s. coverings of $\mathscr{X}_s$.

Claim 2. There exists an isomorphism

$$\tilde{\pi}_1(\mathscr{X}_s)\simeq\pi_1(\Gamma).$$

The proof of Claim 1 is similar to that of Proposition 2.2: The main point to prove is the fact that any finite Galois c.s. covering of X extends to an etale covering of $\mathscr{X}$, and this fact was proved in the proof of Theorem 7.1(1).

The proof of Claim 2 is similar to Theorem 2.4, and left to the readers.

EXAMPLE 7.2 (Mumford curves). Let k be a local field of characteristic 0, and F be the residue field of k. Let $\mathrm{PGL}(2, k)$ denote the two by two projective linear group over k. A discrete subgroup Σ of $\mathrm{PGL}(2, k)$ is called a *Schottky group* if

(i) Σ is finitely generated,

(ii) Σ has no elements $(\neq 1)$ of finite order.

Here we recall the result of Mumford [19]: There exists a one-to-one correspondence between

(a) conjugacy classes of Schottky groups Σ in $\mathrm{PGL}(2, k)$,

(b) isomorphism classes C_Σ of curves C over k, which are the generic fibers of normal schemes $\mathscr{C}$ over $\mathrm{Spec}\, \mathcal{O}_k$ whose special fibers are "F-split degenerate curves," by which we mean 1-dimensional scheme $\mathscr{C}_s$ over F such that

(1) $\mathscr{C}_s$ is reduced.

(2) All irreducible components of $\mathscr{C}_s$ are smooth rational curves over F.

(3) $\mathscr{C}_s$ has as singularities only ordinary double points which are all F-rational.

Moreover, we have the following facts (cf. [8]):

(A) Let C_p be the completion of the algebraic closure of k with respect to the valuation extended from k. Then the set of C_p-valued points of the curve C_Σ has a representation Ω/Σ as a rigid analytic space, where Ω is the set of "ordinary points" of Σ. (It is an open set of $\mathbb{P}^1(C_p)$.)

(B) There exists a tree Δ on which Σ acts freely such that the graph of $\mathscr{C}_s$ has an expression Δ/Σ. In particular Σ is free.

(C) The genus of the curve C_Σ is equal to the rank of Σ.

Remark. The fact that Σ is free was proved first by Ihara [9] by group theretical method. Moreover he computed the rank of Σ using some invariants attached to Σ.

Now, for a Schottky group Σ, we denote by K_Σ the function field of C_Σ. First, from (A), we can see that for a subgroup Σ' of finite index in Σ, there is a finite c.s. covering $f_{\Sigma/\Sigma'}\colon C_{\Sigma'} \to C_\Sigma$.

Conversely, if we are given a finite c.s. covering

$$f\colon C' \to C_\Sigma,$$

then by Theorem 7.1, there exists a subgroup Σ' of Σ, and an isomorphism $\varphi\colon C' \xrightarrow{\sim} C_{\Sigma'}$, such that the following diagram commutes,

$$\begin{tikzcd} C' \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "f"'] & & C_{\Sigma'} \\ & C_{\Sigma} \arrow[ur, "f_{\Sigma/\Sigma'}"'] & \end{tikzcd}$$

In other words, we have an isomorphism

$$\mathrm{Gal}(\tilde{K}_{\Sigma}/K_{\Sigma}) \simeq \hat{\Sigma}.$$

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to Professor Y. Ihara for his unceasing encouragement and his suggestions about Chapter II, Theorem 6.2 and Example 7.2. He also wishes to express his hearty thanks to Professor K. Kato for his valuable advice, without which these researches would have been impossible. In particular, the proof of the reciprocity law (Chap. I, Theorems 1.1 and 2.1) is due to him.

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